A Polynomial-time Bicriteria Approximation Scheme for Planar Bisection

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April 28, 2015

Abstract

Given an undirected graph with edge costs and node weights, the minimum bisection problem asks for a partition of the nodes into two parts of equal weight such that the sum of edge costs between the parts is minimized. We give a polynomial time bicriteria approximation scheme for bisection on planar graphs.

Specifically, let W be the total weight of all nodes in a planar graph G. For any constant $\varepsilon > 0$, our algorithm outputs a bipartition of the nodes such that each part weighs at most $W/2+\varepsilon$ and the total cost of edges crossing the partition is at most $(1+\varepsilon)$ times the total cost of the optimal bisection. The previously best known approximation for planar minimum bisection, even with unit node weights, was $O(\log n)$. Our algorithm actually solves a more general problem where the input may include a target weight for the smaller side of the bipartition.

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1 Introduction

Breaking up is hard to do. The most famous hard graph-breaking problem is *graph bisection*: partitioning the vertices of a graph into two equal-size subsets so as to minimize the number of edges between the subsets. This problem was proved NP-hard by Garey, Johnson, and Stockmeyer [12] in 1976.

Background But how hard is it really? In particular, how well can graph bisection be approximated in polynomial time? Even assuming $P \neq NP$, we cannot at this point rule out the existence of a polynomial-time approximation scheme (PTAS) for graph bisection. The best approximation ratio known to be achievable in polynomial time is $O(\log n)$ where n is the number of vertices, due to Räcke [24], improving on a bound of $O(\log^{1.5} n)$ due to Feige and Krauthgamer [10] (the first result discovered that had a polylogarithmic approximation ratio).

One way to make the problem easier is to relax the balance condition. Given a number $0 < b \leq 1/2$, a bipartition $U \cup V$ of a graph's vertices is *b-balanced*[1] if $|U| \geq \lfloor bn \rfloor$ and $|V| \geq \lfloor bn \rfloor$. Any bipartition $U \cup V$ induces a *cut*, namely the set of edges between U and V. The bisection problem is to find a minimum $\frac{1}{2}$-balanced cut. It might be a simpler problem to find a nearly optimal b-balanced cut for some $b < 1/2$. No better approximation ratio is known for this problem when the input graph is arbitrary. However, for the special case of *planar* graphs, a 2-approximation algorithm was given by Garg, Saran, and Vazirani [13] for finding a minimum b-balanced cut for any $b \leq 1/3$.

Bicriteria approximation[2] gives another way to relax the balance condition. A bicriteria approximation algorithm seeks a b'-balanced cut whose size is at most some factor times the minimum size of a b-balanced cut. In an early and very influential paper, Leighton and Rao [19] showed, using a reduction to their $O(\log n)$-approximation algorithm for another problem, *uniform sparsest cut*, that a b'-balanced cut could be found whose size is $O(\frac{\log n}{b-b'})$ times the minimum size of a b-balanced cut for any $b' < b$ and $b' < 1/3$. Using the improved $O(\sqrt{\log n})$-approximation algorithm of Arora, Rao, and Vazirani [2] for uniform sparsest cut, the approximation ratio $O(\frac{\log n}{b-b'})$ can be improved to $O(\frac{\sqrt{\log n}}{b-b'})$. Note that this performance ratio gets worse as the graph size grows and gets worse as the balance b' achieved by the algorithm approaches the balance b that defines the optimum.

Our results In this paper, we give a *bicriteria approximation scheme* for bisection in planar graphs:

Theorem 1.1. *For any $\epsilon > 0$, there is a polynomial-time algorithm that, given a planar graph G, returns a $\frac{1}{2} - \epsilon$-balanced cut whose size is at most $1 + \epsilon$ times the optimum bisection size.*

That is, the algorithm returns a partition of the vertex set that is *almost* perfectly balanced (each side has at most a fraction $\frac{1}{2} + \epsilon$ of the vertices) and whose size is almost as small as the smallest bisection.

Previously *no approximation algorithm was known that had a $1 + \epsilon$ approximation ratio* even if the algorithm was allowed to return a b'-balanced cut for some constant $b' > 0$, even if b' was allowed to depend on ϵ, even for planar graphs. The algorithm generalizes to handle b-balanced cuts:

[1]Some papers use this term to mean that each part has cardinality *at most* bn.
[2]This approach is also called a pseudo-approximation.

Theorem 1.2. *For any $\epsilon > 0$, there is a polynomial time algorithm that, given a planar graph G and given $0 < b \leq \frac{1}{2}$, returns a $(b - \epsilon)$-balanced cut whose size is at most $1 + \epsilon$ times the minimum b-balanced cut.*

Also, the algorithm can handle nonnegative costs on edges and nonnegative weights on vertices. In fact, we prove a more powerful theorem. Let G be a graph with vertex-weights and edge-costs. For a number b, a *b-bipartition* of G is a bipartition $U \cup V$ of the vertices of G such that the total weight of U is exactly b times the total weight of G. The *cost* of the bipartition $U \cup V$ is the cost of the corresponding cut, i.e. the set of edges connecting U and V. For example, a minimum bisection is a minimum-cost $\frac{1}{2}$-bipartition where all costs and weights are 1. Let $\mathrm{OPT}_b(G)$ be the cost of an optimal b-bipartition.

Theorem 1.3. *For any $\epsilon > 0$, there is a polynomial-time algorithm that, given $b \geq 0$ and given a planar graph G with edge costs and vertex weights such that G has a b-bipartition, returns a b'-bipartition whose cost is at most $(1 + \epsilon)\mathrm{OPT}_b(G)$, where $b' \in [b - \epsilon, b + \epsilon]$.*

1.1 Related work

There is much prior work on finding approximately optimal separators in planar graphs. Before the work of Leighton and Rao [19], Rao [25, 26] gave approximation algorithms for balanced cuts in *planar* graphs. One is a true approximation algorithm for finding a b-balanced cut (for $b \leq \frac{1}{3}$) whose performance guarantee is logarithmic. Another is a bicriteria approximation algorithm whose performance guarantee grows as the balance b' achieved by the algorithm approaches the balance b that defines the optimum. In our algorithm, we employ a subroutine of Rao [26].

Park and Phillips [23] gave improved algorithms for achieving some of the goals of [25, 26]. The aforementioned result of Saran, Garg, and Vazirani [13], the 2-approximation algorithm for b-balanced cut in planar graphs for $b < 1/3$, built on the work of Park and Phillips.

There has been work on approximation schemes for other graph classes. Arora, Karger, and Karpinski [1] gave an approximation scheme for bisection in unit-edge-cost *dense* graphs, graphs with $\Omega(n^2)$ edges. Guruswami, Makarychev, Raghavendra, Steurer and Zhou [14] gave an algorithm that, given a graph in which there is a bisection that cuts a fraction $1 - \epsilon$ of the edges, finds a bisection that cuts a fraction $1 - g(\epsilon)$ of the edges, where $g(\epsilon) = O(\sqrt[3]{\epsilon}\log(1/\epsilon))$.

There has been much work on using approximation algorithms for balanced separators to obtain approximation algorithms for other problems; see the survey of Shmoys [28].

There has been much work on finding balanced separators of size $O(\sqrt{n})$ in planar graphs, regardless of the optimum bisection size. Lipton and Tarjan [20] first gave such an algorithm. Many papers built on this result: improvements to the multiplicative constant, an algorithm to find separators that are also simple cycles [21], algorithms that build on geometric embeddings [22] or eigenvectors [29]. In addition, there has been much work on algorithms that use planar separators.

2 Overview

We outline the algorithm for Theorem 1.3, the bicriteria approximation scheme for min b-bipartition. The input is a graph G^* with vertex weights and edge costs. The algorithm involves edge *contraction*. Contracting an edge uv means removing the edge and replacing its endpoints u and v with a single vertex x. Edges previously incident to u or v are now incident to x. The weight assigned to x is the sum of the weights of u and v.

2.1 Framework for approximation scheme

The algorithm uses a framework of Klein [18]. The framework has previously been used to address optimization problems in planar graphs [4–8, 18], such as traveling salesman and Steiner tree, that involve minimizing the cost of a set of edges subject to *connectivity* constraints. The framework has never been used before in the context of *weight* constraints on the vertices.

First we give the outline. Fix $0 \leq b \leq 1$ and $\epsilon > 0$. The framework uses the notion of *branchwidth*[3] [18, 27].

1. *Spanner step:* In the input graph G^*, contract a selected set of edges, obtaining a graph $\widehat{G^*}$ with the following properties:
   - $c(\widehat{G^*}) \leq \rho \text{OPT}_b(G^*)$, and
   - there exists $b' \in [b-\epsilon, b+\epsilon]$ such that $\text{OPT}_{b'}(\widehat{G^*}) \leq (1+\epsilon)\text{OPT}_b(G^*)$.

   where ρ is a quantity that depends on the construction.

2. *Thinning step:* Select a set S of edges in $\widehat{G^*}$ such that:
   - $c(S) \leq (1/k)c(\widehat{G^*})$ and
   - $\widehat{G^*} - S$ has branchwidth $O(k)$,

   where $k = \epsilon^{-1}\rho$.
3. *Dynamic-programming step:* Find a cheapest b'-bipartition $(\widehat{U_1}, \widehat{U_2})$ in $\widehat{G^*} - S$, where $b' \in [b-\epsilon, b+\epsilon]$.
4. *Lifting step:* Return (U_1, U_2) where U_i is the set of vertices of G^* coalesced to form vertices in $\widehat{U_i}$.

The cost of the returned solution is at most

$$\begin{aligned} \text{OPT}_{b'}(\widehat{G^*} - S) + c(S) &\leq \text{OPT}_{b'}(\widehat{G^*}) + c(S) \\ &\leq (1+\epsilon)\text{OPT}_b(G^*) + (1/k)c(\widehat{G^*}) \\ &\leq (1+\epsilon)\text{OPT}_b(G^*) + \epsilon\rho^{-1}c(\widehat{G^*}) \\ &\leq (1+\epsilon)\text{OPT}_b(G^*) + \epsilon\text{OPT}_b(G^*) \end{aligned}$$

which shows that the cost is at most $1+2\epsilon$ times the cost of an optimal b-bipartition. The fact that $(\widehat{U_1}, \widehat{U_2})$ is a b'-bipartition of $\widehat{G^*}$ means that (U_1, U_2) is a b'-bipartition of G^*.

The thinning step is straightforward: choose a root, and find breadth-first-search levels for all edges in $\widehat{G^*}$. For $i = 0, 1, 2, \ldots, k-1$, let S_i be the set of edges whose levels are congruent mod k to i. For each i, $\widehat{G^*} - S_i$ has branchwidth $O(k)$ (see, e.g., [18], also treewidth $O(k)$, see, e.g., [3]) and at least one of the sets S_i has cost at most $(1/k)c(\widehat{G^*})$.

The fact that $\widehat{G^*} - S$ has branchwidth $O(k)$ means that in the dynamic-programming step an optimal b'-bipartition can be found in time $2^{O(k)}\text{poly}(n, W)$ where W is the sum of weights.

Assume for now that W is $O(n)$ and that ρ is $O(\log n)$. Then the dynamic-programming step requires only polynomial time. (This is a straightforward generalization of Theorem 4.2 of [16].)

The one challenging step is the *spanner step.*[4] The main work of this paper is showing that this step can be done.

[3] *Treewidth* could be used instead

[4]This is usually the case in applications of the framework.

Theorem 2.1. *There is a constant c and a polynomial-time algorithm that, given $\epsilon > 0$, $b > 0$ and a planar embedded graph G^* with vertex weights and edge-costs, returns a graph $\widehat{G^*}$, obtained from G^* by contracting edges, with the following properties: $c(\widehat{G^*}) \leq c \log n \cdot \mathrm{OPT}_b(G^*)$, and $\exists b' \in [b-\epsilon, b+\epsilon]$ such that $\mathrm{OPT}_{b'}(\widehat{G^*}) \leq (1+\epsilon)\mathrm{OPT}_b(G^*)$.*

Once we have proved Theorem 2.1, showing that there is a poly-time algorithm for the *spanner* step of the framework, we will have proved Theorem 1.3.

2.2 Spanner construction overview

Many tools have been developed for spanner construction. One tool first used for Steiner TSP is this *boundary-to-boundary spanner*:

Lemma 2.2 (Theorem 6.1 of [17]). *Let G be a planar embedded graph with edge-costs, and let C be the boundary of some face of G. For any $\epsilon > 0$, there is a subgraph H of cost $O(\epsilon^{-4}c(C))$ such that, for any two vertices x and y of C, the x-to-y distance in H is at most $1+\epsilon$ times the x-to-y distance in G. Furthermore, there is an $O(n \log n)$ time algorithm to derive H from G.*

The edges defining the min-cost bisection or b-bipartition of input graph G^* correspond in the planar dual G to a collection of edge-disjoint cycles. Fragments of these cycles are paths; perturbing the solution by replacing such a path with a nearly shortest path in a subgraph H does not increase the cost of the solution by much. This simple idea is at the heart of the spanner construction.

At the highest level, we use the following strategy: (Step 1) select a collection of cycles, (Step 2) join some of them together with paths, (Step 3) for each region of the planar dual bounded by these cycles and paths, for each cycle C that forms part of the boundary of that region, construct the boundary-to-boundary spanner for C-to-C paths in that region. The union of edges from Steps 1, 2, and 3 form the spanner.

So far, however, we have not handled weights. Indeed, a perturbation (in which a path P of the optimal solution is replaced with a path P' in the spanner) could shift weight from one side of the bipartition to the other. We need a way to limit the amount of weight that could shift. This is the purpose of Step 1. Note that the original path P and replacement path P' form a cycle C, and that weight that could shift is enclosed by C. The goal of Step 1 is to ensure that, for every such cycle C derived from such a perturbation, the weight enclosed by C is small compared to the cost of C. That way, the total amount of weight shifted can be charged to the cost of the optimal solution.

Step 1 ensures that such cycles' cost/weight ratios are large by greedily finding a collection of mutually noncrossing cycles of small ratio. Once Step 1 has completed, each of the regions bounded by the noncrossing cycles contains no cycle with small cost/weight ratio (essentially).

The fact that cycles found in Step 1 have small cost/weight ratio is used to show that the total cost of all those cycles is not much more than the cost of the optimal solution. Each cycle's cost is charged to the weight of *some* of the faces it encloses. If we ensure that each face is charged at most a logarithmic number of times, the total cost of the cycles is at most a log times OPT.

To ensure logarithmic charging, Step 1 alternates between adding cycles to the spanner and removing cycles. When one cycle is enclosed by the other but the two cycles enclose almost the same weight, the pair of cycles is designated a *splicing pair*, and, in an operation called *splicing*, cycles sandwiched between them are removed from the spanner.

Here is one complication: A region bounded by cycles from Step 1 often is bounded by *several* cycles, i.e. its boundary is disconnected. The boundary-to-boundary spanner (Lemma 2.2) works only for a connected boundary. Step 2 therefore uses a technique called *PC clustering*, due to

Bateni, Hajiaghayi, and Marx [6], to add paths joining some of the boundary components. If two boundary components remain unconnected after PC clustering, we can assume that some near-optimal solution does not connect them.

Here is another complication: Consider a cycle C associated with a perturbation that replaces a path P of the optimal solution with a path P' in the spanner. If C happens to be sandwiched between two cycles comprising a splicing pairs, then C might have small cost/weight ratio. This happens if C encloses the inner cycle C' of the splicing pair. To make sure no such cycle C is used in a perturbation, Step 3, in forming the boundary-to-boundary spanners for the region R containing C', distinguishes between paths going clockwise around C' and paths going counterclockwise. This is accomplished using a construction from topology, the *cyclic double cover*.

3 Preliminaries

Achieving polynomially bounded weights Garg, Saran and Vazirani [13] observed that if one is willing to accept a $(b \pm \varepsilon)$-bipartition instead of a b-bipartition, then one can assume polynomially bounded weights. This is done by defining new weights $w(v) \leftarrow \lfloor w(v) \frac{n}{\varepsilon W} \rfloor$ where W is the sum of original weights and n is the number of vertices of the input graph. After the transformation, the sum of weights is bounded by $\varepsilon^{-1} n$. However, due to the truncation, the weight of any vertex may be off by $\frac{\varepsilon W}{n}$ with respect to its original weight. Therefore, the weight of any set in a bipartition may be off by at most εW. This is allowed by our theorems. We therefore assume henceforth that the sum of weights is polynomially bounded.

Basic definitions We assume that the reader is familiar with the basic concepts of planar graphs such as planar embeddings and planar duality. We use G^* to denote the planar embedded input graph, and we use G to denote its planar dual. The costs of edges in G^* are assigned to the corresponding edges of G. The weight function can be viewed as an assignment of weights to faces. For the remainder of this paper we deal with the dual graph G. Henceforth, unless otherwise stated, vertices, faces and edges refer to those of G.

For each edge e in the edge-set E, we define two oppositely directed darts, one in each orientation. We define $rev(\cdot)$ to be the function that maps each dart to the corresponding dart in the opposite direction.

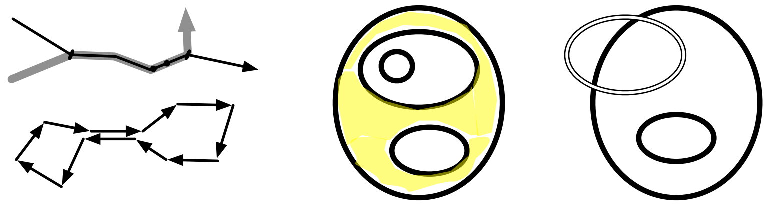
A non-empty sequence $d_1 \ldots d_k$ of darts is a *walk* if the head of d_i is the tail of d_{i+1} for every $1 \leq i \leq k$. A walk is said to be a *closed walk* if the tail of d_1 is the head of d_k.

Let X be a walk in a planar embedded graph, and let $P = a\ X\ b$ and $Q = c\ X\ d$ be walks that are identical except for their first and last darts. Let a' be the successor of a in P and let b' be the predecessor of b in P. We say Q forms a *crossing configuration* with P if the clockwise cyclic order of darts whose head is $head(a)$ induces the cycle $(a\ rev(a')\ c)$ and the clockwise cyclic order of darts whose tail is $tail(b)$ induces the cycle $(b\ rev(b')\ d)$.

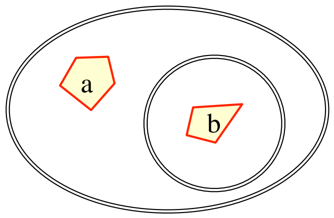
We say a walk P *crosses* a walk Q if a subwalk of P and a subwalk of Q form a crossing configuration. See Figure 1(a). We define a *cycle (of darts)* to be a non-self-crossing closed walk that uses each dart at most once. We omit the modifier "of darts." Thus for our purposes a cycle can use an edge at most twice–once in each direction—and cannot cross itself.

We can assume (by adding a self-loop if necessary) that G has a face with zero weight whose boundary has zero cost. We use f_∞ to denote this face, and we refer to it as the infinite face.

Let C be the dart multiset of a set of closed walks, and let C^* be the multiset of corresponding darts in G^*. Let f be a face of G. Let P be any f_∞-to-f path in G^* (the input graph). We say f is *enclosed* by C if the parity of $|P \cap C^*| - |P \cap rev(C^*)|$ is odd. See Figure 1(b).



(a) Two crossing paths and a cycle of darts (left), a set of mutually noncrossing cycles (middle), and a set of cycles that are not mutually noncrossing (right). In the middle, the region bounded by three of the cycles is shaded.



(b) A set $\mathcal{C}$ of two cycles (double lines). Face a is enclosed by $\mathcal{C}$, but face b is not.

Figure 1. Crossings and enclosure

Consider a bipartition $U \cup V$ in G^* where the vertex of G^* corresponding to f_∞ belongs to V. The duality of cuts and cycles implies that the edges crossing this cut form a set $\mathcal{C}$ of cycles in G. The weight of the bipartition in G^* is the total weight of faces enclosed by $\mathcal{C}$.

For a cycle C of G, we define $w_G(C)$ to be the sum of weights of faces enclosed by C in G. For a set S of faces of G, we define $w_S(C)$ to be the sum of weights of faces in S enclosed by C, and we define $r_S(C)$ to be the ratio of the cost of C to $w_S(C)$.

Regions defined by a mutually non-crossing set of cycles Let $\mathcal{C}$ be a set of cycles that are mutually non-crossing. See Figure 1(a). Assume for the sake of convenience that the boundary of f_∞ is in $\mathcal{C}$. The set $\mathcal{C}$ can be represented by a rooted ordered tree $\mathcal{T}$. Every node $v \in \mathcal{T}$ corresponds to a cycle $C_v \in \mathcal{C}$. Ancestry in $\mathcal{T}$ is determined by enclosure: Node v is an ancestor of node u in $\mathcal{T}$ if C_v encloses C_u in G. Thus, f_∞ is the cycle corresponding to the root of $\mathcal{T}$.

Every node v of $\mathcal{T}$ is associated with a *region* R_v. R_v is the subgraph of G consisting of vertices, edges, and faces enclosed by C_v, and not strictly enclosed by C_u for any child $u \in \mathcal{T}$ of v.

The cycle C_v is called the *outer boundary* of R_v. For a child u of v, the cycle C_u is called a *hole* of R_v. The *weight* of a hole is the total weight of faces enclosed by the hole. Together, the outer boundary and the holes form the *boundary* of R_v. We say R_v *strictly contains* an edge if in addition the edge does not belong to the boundary of R_v.

We say that a region R *contains* a cycle C if R contains every edge of C, and that R *strictly contains* C if in addition R strictly contains at least one edge of C.

Finding low-ratio cycles Let T be a shortest path tree with root r. Let C be a cycle that encloses r. We say that C is discovered by T from the inside if, for every $v \in C$, the r-to-v path in T is enclosed by C. Rao [26] described a polynomial time technique that finds, for every possible weight w, the minimum-cost cycle enclosing exactly w weight among cycles that go through r and are discovered by T from the inside (if such a cycle exists).

The following lemma implies that Rao's technique can be used to find a maximally face-enclosing cycle with ratio at most some ratio α.

Lemma 3.1. *Let G be a planar embedded graph with edge costs and face weights. Let T be a shortest path tree in G, rooted at a vertex r. Let C be a maximally face-enclosing cycle C with ratio at most α. If C encloses r then T discovers C from the inside.*

Proof: Assume that C is not discovered by T from the inside. Then there is a vertex $v \in C$ such that the r-to-v path P in T is not enclosed by C. Let P' be a maximal subpath of P consisting only of edges that are not enclosed by C. Let x, y be the endpoints of P'. Note that, since C encloses r, both x and y are vertices of C. Let Q' be a subpath of C with endpoints x and y such that the cycle $C' = Q' \circ P'$ encloses C. Since P' is a shortest x-to-y path, the cost of C' is at most the cost of C. Cycle C' also encloses every face enclosed by C, so the weight of C' is at least the weight of C. This contradicts the fact that C is a maximally face-enclosing cycle C with ratio at most α. □

To find a maximally face enclosing cycle with ratio at most α, consider the set M of cycles whose weight is maximum among all cycles whose ratio is at most α. Let C be a cycle in M enclosing the greatest number of faces (there may be faces with zero weight). Note that C is a maximally face enclosing cycle with ratio at most α. Let w denote the weight of C. To find C, slightly perturb the weight of every face to make it non-zero without significantly changing the total weight of any cycle. This can be done by scaling the weights by the number of faces, and adding 1 to the weight of every face. Note that this transformation keeps the total weight polynomially bounded. For every possible choice of root r of the shortest path tree T, use Rao's technique to compute, for each possible (perturbed) weight x, the minimum cost cycle discovered from the inside, and enclosing exactly x perturbed weight. Return the cycle enclosing maximum perturbed weight whose unperturbed ratio is at most α.

Let w' be the perturbed weight of C. By Lemma 3.1, for a correct choice of r, C is the minimum-cost cycle computed for weight w'. By definition of the perturbation, the perturbed weight w' of C is greater than that of any other cycle with ratio at most α that encloses fewer faces than C. Also, no cycle whose (perturbed) weight is greater than w' has ratio at most α with respect to unperturbed weights. Therefore, the procedure described returns the cycle C.

Edge contractions One step of the algorithm described in this paper uses edge contractions. The planar embedding is not relevant in this step. We therefore use a definition of edge contraction that does not depend on the embedding. Contracting an edge e results in merging the endpoints of e into a single vertex. Any self loops that are created in the process are deleted.

4 Skeleton construction

We now begin to describe our procedure for constructing a spanner to approximate a minimum cost b-bipartition. For brevity, we let OPT be the cost of an optimal b-bipartition. Let W be the total weight of all faces in G. Recall that we work directly with the dual graph G of our input graph G^* and that a solution or b-bipartition is a set of cycles in G. We assume that the spanner algorithm is given a rational number λ such that $\mathrm{OPT}/W \leq \lambda \leq 2\mathrm{OPT}/W$, since an outer loop surrounding the approximation algorithm can try different values of λ and return the best solution obtained.

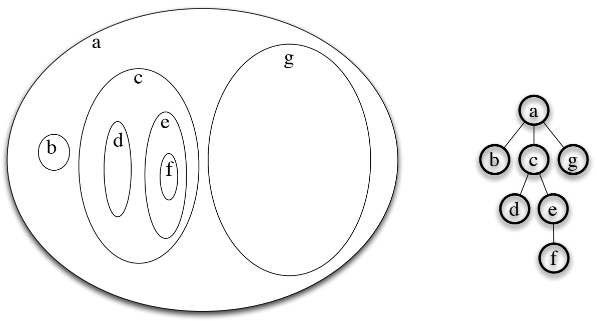


Figure 2. On the left are non-crossing cycles. On the right is the corresponding tree. Siblings are ordered left to right in increasing order of weight enclosed. The cycles are labeled in preorder.

The algorithm constructs a family $\mathcal{C}$ of mutually noncrossing cycles, which includes the outer face f_∞. We refer to $\mathcal{C}$ as the *skeleton*. As discussed in Section 3, the cycles of $\mathcal{C}$ define *regions*, and define a rooted tree $\mathcal{T}$ based on enclosure. For each node v of this tree, we order the children of v left to right according to increasing weight of the faces they enclose. See Figure 2. A preorder traversal of $\mathcal{T}$ that visits siblings in this order defines a total left-to-right ordering on the nodes of $\mathcal{T}$.

The skeleton building algorithm is given as Algorithm 1. As the algorithm progresses, cycles and regions are defined with respect to the *current* set of cycles $\mathcal{C}$.

Algorithm 1 Skeleton(G, λ)

```
 1: $\mathcal{C} \leftarrow \{$boundary of $f_\infty\}$
 2: ptr $\leftarrow$ *null*
 3: **repeat**
 4:   **if** some region $R$ with respect to $\mathcal{C}$ strictly contains a cycle $C$ s.t. $r_R(C) \leq \epsilon^{-1}\lambda$ **then**
 5:     let $C$ be a **maximally** face-enclosing such cycle
 6:     set $F(C)$ to be the set of faces of $R$ enclosed by $C$
 7:     add $C$ to $\mathcal{C}$
 8:   **else**
 9:     let the cycles of $\mathcal{T}$ in preorder be $C_1 \ldots C_k$
10:     **if** ptr $=$ *null* **then** ptr $\leftarrow C_k$
11:     **else** ptr $\leftarrow$ cycle preceding ptr in preorder of $\mathcal{T}$
12:     let $C_q$ be the value of ptr
13:     let $p$ be min such that $C_p$ is an ancestor of $C_q$ and $w_G(C_p) < 2\,w_G(C_q)$
14:     remove cycles $C_{p+1}, \ldots, C_{q-1}$ from $\mathcal{C}$
15: **until** ptr $=$ boundary of $f_\infty$
16: return $\mathcal{C}$
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Consider an execution of Line 14 where a sequence of cycles $C_{p+1}, \ldots, C_{q-1}$ is removed from $\mathcal{C}$. We say that (C_p, C_q) is a *splicing pair*. Cycle C_p is the *rootward* member of the pair, and C_q is the *leafward* member.

The following lemmas establish several useful properties of the skeleton, including the polynomial running time of the skeleton-building algorithm.

Lemma 4.1. *Let B be the value of* ptr *at any time t in which Line 12 is executed. The suffix of the preorder starting at B does not change after time t. In particular, cycle B will never be removed from the skeleton.*

Proof: The proof is by induction on the number of times line 12 is executed after time t. If this number is zero, i.e. if time t is the last execution of line 12, then $\mathcal{T}$ can only change when a splice with splicing pair (A, B) occurs, for some cycle A. By definition of the algorithm, no new cycles can be added to $\mathcal{T}$ at time t. Splicing with splicing pair (A, B) does not change the region of any cycle to the left of A (exclusive) or to the right of B (inclusive). Therefore, no new cycles can be added to the left of A or to the right of B after the splice. In other words, any new cycle C added after the splice is added as a descendent of A, but not of B. Since $w_G(B) > \frac{1}{2}w_G(A)$, B always lies on the rightmost path in the subtree of $\mathcal{T}$ rooted at A. Hence any such new cycle C is added left of B in the preorder.

For the inductive step, by the argument above, the suffix starting at B does not change until the next time line 12 is executed. Since `ptr` moves strictly to the left, the inductive hypothesis implies that the suffix remains unchanged for the remainder of the execution. □

Lemma 4.2. *The skeleton algorithm runs in polynomial time.*

Proof: By Lemma 4.1, each time line 12 is executed, the length of the fixed suffix of the preorder of $\mathcal{T}$ increases by one. The number of cycles in the skeleton is bounded by the number of faces in G, so there are $O(n)$ splicing steps. Hence, the total number of cycles added to $\mathcal{C}$ throughout the execution of the skeleton construction algorithm is $O(n^2)$. Since each cycle is found in polynomial time, the total running time is polynomial. □

Lemma 4.3. *The cost of the skeleton is* $O(\varepsilon^{-1}\mathrm{OPT}\log W)$.

Proof: Let f be a face of G, and let $C(f) = \{C \in \mathcal{C} : f \in F(C)\}$. By Lemma 4.1, every cycle A in the final skeleton is pointed to by `ptr` at some time t. Cycle A then participates in a (possibly trivial) splice, removing all but one ancestor cycle with weight less than twice $w_G(A)$. Afterward, for each cycle C added to $\mathcal{C}$, we have $F(C) \cap F(A) = \emptyset$. Let $C_1, \ldots, C_k$ be all cycles of $C(f)$ in rootward ordering. The weight of the cycles in the sequence doubles at least once every other cycle. We have $|C(f)| = O(\log W)$.

The ratio of each cycle C in the skeleton is bounded by $\varepsilon^{-1}\lambda$, so $c(C) \le \varepsilon^{-1}\lambda w_G(F(C))$. Thus, the total cost of cycles in the skeleton is at most

$$\begin{aligned}\sum_C \varepsilon^{-1}\lambda w_G(F(C)) &\le \varepsilon^{-1}\lambda \sum_{f\in G} |C(f)| \cdot w(f)\\ &\le O(\varepsilon^{-1}\lambda W \log W)\\ &= O(\varepsilon^{-1}\mathrm{OPT}\log W).\end{aligned}$$

□

The heavy nesting of low-ratio cycles in the skeleton guarantees the following lemma, which will be crucial in proving our spanner construction is correct.

Lemma 4.4. *Suppose that, when the algorithm terminates, a region R contains cycle $C \notin \mathcal{C}$, and $r_G(C) \le \varepsilon^{-1}\lambda$. Then C encloses the heaviest hole of R, and $r_S(C) > \epsilon^{-1}\lambda$ where $S = \{$faces enclosed by outer boundary of R but not by heaviest hole$\}$*

Proof: We say a cycle D *weakly crosses* C if D crosses C or $D = C$. The following claim is immediate:

Claim 1: At termination, $\mathcal{C}$ contains no cycle that weakly crosses C.

Claim 2: Some splice removes a cycle weakly crossing C.

Proof of Claim 2: At some point the algorithm adds to $\mathcal{C}$ a cycle that weakly crosses C or is strictly enclosed by C. (If this never happens then C remains available to choose in line 5, and the algorithm is not ready to terminate.) Furthermore, by maximality in line 5, before the algorithm adds to $\mathcal{C}$ any cycle strictly enclosed by C, it must add a cycle that weakly crosses C. By Claim 1, such a cycle must have been removed by a splice. $\square$

Claim 3: For the last splice removing a cycle D that weakly crosses C, the splicing pair (A, B) must satisfy the following condition:

$*$ each edge of C is enclosed by A and not strictly enclosed by B.

Proof of Claim 3: Since the splice removes D, every edge of D is enclosed by A and not strictly enclosed by B. After that splice, $A \in \mathcal{C}$. If C weakly crosses A then by Claim 1 A must be removed later, contradicting the choice of (A, B). Similarly, C does not weakly cross B. This implies every edge of C is enclosed by A and is not strictly enclosed by B, proving the claim.

Claims 2 and 3 imply that there is some splice whose splicing pair satisfies Condition *. Let (A^0, B^0) be the splicing pair of the *last* such splice. Let t_0 be the time of that splice.

Let $S_0 = \{$faces enclosed by A^0 and not by $B^0\}$. We claim $r_{S_0}(C) > \epsilon^{-1}\lambda$. If not then after time t_0 the cycle C would still be available to add to $\mathcal{C}$, so some cycle weakly crossing C would have been added, contradicting either Claim 1 or the definition of t_0. Because $r_S(C) > r_G(C)$ and B^0 strictly contains no edge of C, it follows that C encloses B^0. By Lemma 4.1, cycle B^0 is never subsequently removed from $\mathcal{C}$. Also, by Claims 1 and 3 no cycle weakly crossing C is subsequently added to $\mathcal{C}$. Therefore, for every $t \geq t_0$, at time t there is a unique pair A^t, B^t of cycles in $\mathcal{C}$ such that A^t encloses C, and C encloses B^t, and A^t is the parent of B^t in the tree $\mathcal{T}$ of cycles of $\mathcal{C}$.

Since B^0 remains in $\mathcal{C}$, we infer B^t encloses B^0 for all $t \geq t_0$. We show by induction that at each time $t \geq t_0$, cycle A^0 encloses A^t. There are two mutually exclusive cases. First, suppose that just after time t a cycle A^{t+1} enclosing C is added to $\mathcal{C}$ and becomes the parent of B^t. Then, since A^t is still in $\mathcal{C}$, A^t encloses A^{t+1} and A^0 encloses A^{t+1}.

Second, we show that no splice at time $t+1$ can remove A^t. Assume for a contradiction that some splice did remove A^t, and let (A, B) be the splicing pair. If B were not a descendant of A^t in $\mathcal{T}$ then the fact that the splice removes A^t would mean it would remove all descendants of A^t, including B^0, a contradiction. If B crosses C then by Claim 3 the splice or a later one satisfies Property *, contradicting the choice of t_0. Therefore B is a proper descendant of A^t that does not cross C. B cannot enclose C else it would be a parent of B^t, contradicting the fact that A^t, B^t are a parent-child pair. Therefore B strictly encloses no edge of C. But then (A, B) satisfy Property *, a contradiction.

Let T be the time the algorithm terminates. We have shown that A^0 encloses A^T and B^T encloses B^0. Let $S = \{$faces enclosed by A^T and not by $B^T\}$. It follows that $w_G(B^T) \geq w_G(B^0)$ and $w_G(A^T) \leq w_G(A^0)$ and $S \subseteq S_0$. Because $w_G(A^0) < 2w_G(B^0)$, we have $w_G(A^T) < 2w_G(B^T)$, so B^T is the heaviest hole of the region whose outer boundary is A^T. Because $S \subseteq S_0$ and $r_{S_0}(C) > \epsilon^{-1}\lambda$, it follows that $r_S(C) > \epsilon^{-1}\lambda$. This completes the proof.

5 Shortcuts

Consider an optimal solution O. Let K be a cycle of O that crosses the skeleton. A *path decomposition* of K is a decomposition of K into paths $p_0, p_1, \ldots$ such that the number of paths is minimized and no path p_i crosses the skeleton. Because the number of paths is minimized, the endpoints of the paths occur only on skeleton cycles that are crossed by K.

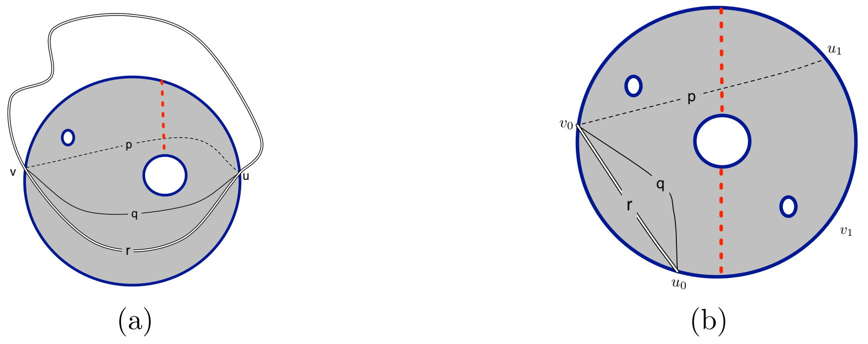

Figure 3. (a) Path r of a path decomposition enters and leaves a region at vertices u and v. Replacing r with shortcut q will only affect the enclosure of faces outside the largest hole of the region. Replacing r with shortcut p will affect faces within the hole as well. (b) The cyclic double cover of the region. Lifts of r and q have the same endpoints in the cyclic double cover, but no lift of p has the same two endpoints. Therefore, p would not be a considered a shortcut for r within the cyclic double cover.

Let p be a member of a path decomposition as defined above. Let u and v be the endpoints of p and let R be the region of the skeleton that contains p. In order to build the spanner, we wish to replace p by a $(1+\varepsilon)$-approximately shortest u, v-path p' in R. For concision, we call the approximately shortest path p' a *shortcut*. The spanner contains a set of edges that carry shortcuts for all paths of all path decompositions. These edges are chosen so their total cost is sufficiently small and replacing each path by its shortcut does not perturb the weight enclosed by the solution by more than a small amount.

If R contains no holes, then it is relatively easy to compute shortcuts within R. Let C be the outer boundary of R. The spanner algorithm computes a subgraph A of G within R using the boundary-to-boundary spanner of Lemma 2.2.

As we show later in Lemma 8.3, replacing paths through regions without holes by their shortcuts does not change the weight of the solution by very much. Essentially, the faces that change sides when a path p is replaced by its shortcut p' are exactly those enclosed by $p \circ rev(p')$. The cycle enclosing these same faces must have low weight or there would exist a low ratio cycle within R violating Lemma 4.4.

Unfortunately, this argument does not hold if R contains holes. First, there may be paths in a path decomposition that start and end on different boundary components of R. A boundary-to-boundary spanner does not contain shortcuts for such paths. And even if shortcuts are available, $p \circ rev(p')$ could enclose a high weight hole, and all of the faces within that hole could change sides after replacing p with p'. See Figure 5 (a). Lemma 4.4 does not imply anything about a cycle's weight if it encloses the largest hole of a region, so there would be no limit to how much weight could change sides. In the next section, we describe how to address both of these issues for regions with holes. We begin by describing a topological construct called the cyclic double cover used to address the later issue. A procedure called PC-clustering will be used to address the former.

6 The cyclic double cover

In this section, we describe a tool called the *cyclic double cover* that is used by our spanner algorithm. Thanks to this tool, the spanner will carry shortcuts that do not force the largest hole in each region to change sides. The cyclic double cover as described here was originally used by Erickson [9] and by Fox [11] to find short topologically interesting cycles in surface embedded graphs. Our presentation of the cyclic double cover is based closely on theirs.

Let R be a region with outer boundary C_0 and largest hole C_i. Let L be an arbitrary path from C_0 to C_i. Cut along L to create a new region R' where C_0, C_i, and two copies of L (L^+ and L^-) form the outer boundary. Let $(R', 0)$ and $(R', 1)$ be two distinct copies of R'. For any vertex v in R, let $(v, 0)$ denote the copy of v in $(R', 0)$ and let $(v, 1)$ denote the copy in $(R', 1)$. Finally, let $(L^\pm, 0)$ denote the copies of $L^\pm$ in $(R', 0)$ and $(L^\pm, 1)$ denote the copies in $(R', 1)$. The cyclic double cover R^2 is the planar graph resulting from identifying $(L^+, 0)$ and $(L^-, 1)$ to a single path $(L, 0)$ and identifying $(L^+, 1)$ and $(L^-, 0)$ to $(L, 1)$. Every hole of R appears twice as a hole in R^2 except for C_i. The edges of C_0 appear twice along the outer face of R^2 and the edges of C_i appear twice along a single hole in R^2. See Figure 5 (b).

The cyclic double cover R^2 has an equivalent combinatorial definition. Each vertex v from R has two copies $(v, 0)$ and $(v, 1)$ in R^2. Edge uv has two copies $(u, 0)(v, 0)$ and $(u, 1)(v, 1)$ if uv does not enter L from the left. Otherwise, the copies of uv are $(u, 0)(v, 1)$ and $(u, 1)(v, 0)$. Edge copies in R^2 retain their costs from R. The *projection* of any vertex, edge, or walk in R^2 is the natural map to R that occurs by dropping the 0 or 1 from the vertex and edge tuples. We say a vertex, edge, or walk p in R *lifts* to p' if p is the projection of p'. The outer boundary of R^2 projects to two copies of C_0 and one boundary of R^2 projects to two copies of C_i. Otherwise, every face or boundary of R^2 projects to a face or hole in R. Call a boundary in R^2 a hole if it projects to one or two copies of a hole in R. For a walk p in R, let $x^2(p) = 0$ if p crosses L an even number of times. Otherwise, let $x^2(p) = 1$. We immediately get the following lemmas.

Lemma 6.1. *Let p be a walk in R from vertex u to vertex v. Walk p is the projection of a unique walk p' in R^2 from vertex $(u, 0)$ to vertex $(v, x^2(p))$.*

Lemma 6.2. *Let C' be a closed walk in R^2. C' projects to a unique closed walk C in R that does not enclose C_i.*

Our algorithm builds a spanner by computing the cyclic double cover for each region R of the skeleton that contains a hole. The construction of the double covers can be done in linear time. It then computes a set of edges within each double cover that carry shortcuts for the lifts of paths that may appear in path decomposition. See Figure 5 (b). The projections of these edges back into the original regions will be added to the spanner. In Lemma 8.4, we will show that we can replace the paths of path decompositions by the projections of these shortcuts without changing the weight of the solution by more than a small amount.

7 PC-clustering

Our spanner algorithm needs to find edges that carry shortcuts within each cyclic double cover. However, the
boundary-to-boundary spanner of Lemma 2.2 will not find edges to carry shortcuts between distinct boundaries of a region. In order to use the algorithm, we augment the skeleton's edges within each double cover using the PC-clustering algorithm of Bateni, Hajiaghayi, and Marx [6].

Let K be a cycle in the optimal solution that crosses the skeleton, and let p be a path of K's path decomposition where p lies in a region R with a hole. The PC-clustering algorithm adds a relatively cheap set of edges to the boundary of the cyclic double cover R^2. These edges are chosen so that, *in general*, both of p's endpoints lie on the same boundary component after running PC-clustering. If p's endpoints are still on different components, then at least one of the components must be very cheap. The edges of that component can be added to the optimal solution without substantially increasing its cost, and K can be modified to avoid crossing the skeleton cycles in that component. Path p is no longer in a path decomposition, and it is no longer necessary for the spanner to hold a shortcut between p's endpoints. Formally, the PC-clustering algorithm can be described as follows.

Lemma 7.1 (Bateni *et al.* [6]). *Let $G(V,E)$ be a graph with non-negative edge costs $c(e)$ and face potentials $\phi(v)$. There exists a polynomial time algorithm to find a subgraph Z such that*

1. *the total cost of Z is at most $2\sum_{v\in V}\phi(v)$ and*
2. *for any subgraph H of G, there is a set U of vertices such that*
   - *(a) $\sum_{v\in U}\phi(v)$ is at most the cost of H and*
   - *(b) if two vertices $v_1, v_2 \notin U$ are connected by H, then they are in the same component of Z.*

For each region R containing a hole, our algorithm does the following: It contracts the lifts of every skeleton edge in R^2 to get the graph $\hat{R}^2$. For any vertex v in $\hat{R}^2$, let $c(v)$ be the total cost of edges contracted to create v (implicitly, if v appears in R^2 as well, then $c(v) = 0$). For each v in $\hat{R}^2$, the algorithm assigns a potential $\phi(v) = \varepsilon^{-1}c(v)$. The algorithm them applies the PC-clustering procedure of Lemma 7.1 to $\hat{R}^2$ to get the set of edges Z.

Let the *well-connected cover graph* be the set of boundary in R^2 unioned with Z. The edges in the well-connected cover graph are the *well-connected cover edges*. The projections of the well-connected cover edges will be used in the spanner.

Lemma 7.2. *The total cost of all well-connected cover edges is $O(\varepsilon^{-2}\mathrm{OPT}\log W)$.*

Proof: By Lemma 4.3, the total cost of all cycles in the skeleton is $O(\varepsilon^{-1}\mathrm{OPT}\log W)$. Every cycle in the skeleton appears on the boundary of at most two regions. For each boundary of a region, each edge of the boundary appears at twice in that region's cyclic double cover. Therefore, the sum of vertex potentials used for PC-clustering across all cyclic double covers is at most $O(\varepsilon^{-2}\mathrm{OPT}\log W)$. The lemma follows from the first property of PC-clustering's output as defined in Lemma 7.1. □

We argue there exists a near-optimal solution such that shortcuts do not start and end on different components of the well-connected cover graphs.

Lemma 7.3. *There exists a solution with cost at most $(1+2\varepsilon)\mathrm{OPT}$ enclosing exactly bW weight such that for each cycle K in the solution,*

1. *either K does not cross the skeleton*
2. *or K has a path decomposition $p_0, p_1, \ldots$ such that a lift of each path p_i in a region with a hole has both endpoints on the same component of that region's well-connected cover graph.*

Proof: Consider the optimal solution O. For each region R with a hole, let O_R be the subset of edges from O that lie strictly within the region R. Consider a lift of O_R to R^2, and let $\hat{O}_R$ be the edges that remain after performing contractions to get $\hat{R}^2$. Let $\hat{U}_R$ be the set of vertices in $\hat{R}^2$ that are guaranteed to exist for $\hat{O}_R$ by the second property in Lemma 7.1.

Each vertex of $\hat{U}_R$ is the result of contracting zero or more lifted skeleton cycles in R^2. Let $\mathcal{U}$ be the set of all skeleton cycles where for each cycle $C \in \mathcal{U}$, the contraction of a lift of C lies in some $\hat{U}_R$. The cycles in $\mathcal{U}$ are mutually non-crossing, so they partition the faces of G into *$\mathcal{U}$-regions*. For each $\mathcal{U}$-region R', let $O_{R'}$ be the boundary of faces in R' enclosed by O (therefore, the holes of R' are not enclosed by $O_{R'}$). Let O' be the union of cycles over all $O_{R'}$. Solution O' encloses the same set of faces of G as O. No cycle of O' crosses a member of $\mathcal{U}$. Edges strictly internal to some $\mathcal{U}$-region are used exactly once and only if they are used in O. Finally, each cycle $C \in \mathcal{U}$ may contribute up to two copies of some of its edges to O', because C lies on the boundary of two $\mathcal{U}$-regions.

The cost difference between O' and O is at most twice the cost of cycles in $\mathcal{U}$. For a single region R, the total potential of vertices in $\hat{U}_R$ is at most the cost of $\hat{O}_R$. By definition of vertex potentials, the cycles of $\mathcal{U}$ that contribute to set $\hat{U}_R$ have total cost at most $\varepsilon c(\hat{O}_R)$. All sets $\hat{O}_R$ are edge-disjoint, so twice the total cost of all cycles in $\mathcal{U}$ is at most $2\varepsilon\mathrm{OPT}$.

Now, consider any cycle K in O' that crosses the skeleton, and let $p_0, p_1, \ldots$ be a path decomposition for K. The endpoints of each path p_i lie on skeleton cycles crossed by K. For any p_i lying in a region R with a hole, let p_i' be a lift of path p_i to R^2. Let $p_i^{/}$ be the path that results from p_i' after contracting edges to make $\hat{R}^2$. Finally, let v_1 and v_2 be the endpoints of $p_i^{/}$. Neither v_1 nor v_2 lie on a member of $\hat{U}_R$ as defined above, because the cycles of O' do not cross any members of $\mathcal{U}$. Further, every edge of p_i' that is not contracted is a member of $\hat{O}_R$. Therefore, Lemma 7.1 guarantees v_1 and v_2 lie in the same component of PC-clustering's output. Each vertex of $\hat{R}^2$ is a connected component of the boundary of R^2 so the endpoints of p_i' lie on the same component of the well-connected cover graph as well. □

7.1 Finding shortcuts

For each cyclic double cover R^2, for each well-connected cover component, our spanner algorithm computes edges carrying shortcuts between every pair of vertices on the component. It does so using the following extension of the boundary-to-boundary spanner of Lemma 2.2. The projection of these edges is added to our spanner.

Lemma 7.4. *Let G be a planar graph with non-negative edge costs $c(\cdot)$. Let A be a component of G. Then for any $\varepsilon > 0$, there is an $O(n \log n)$ time algorithm to compute a subgraph H of G where $c(H) = O(\varepsilon^{-4} c(A))$ and for any pair of vertices u and v on A, $\mathrm{dist}_H(u,v) \le (1+\varepsilon)\mathrm{dist}_G(u,v)$.*

Proof: The edges and vertices of A partition the plane into one or more components. The algorithm cuts the planar graph G along A, separating these components so that the edges of A lie on their boundary. For each boundary cycle C of the cut open graph, the algorithm runs the boundary-to-boundary spanner procedure of Lemma 2.2 to create a subgraph H_C of G of cost at most $O(\varepsilon^{-4} c(C))$ in $O(n_C \log n_C)$ time where n_C is the number of vertices in C's component of the cut open planar graph. Subgraph H is the union of all such subgraphs H_C. Each edge of A appears on boundary twice, so the total cost of all subgraphs H_C is at most $O(\varepsilon^{-4} c(A))$. The cut open surface has $O(n)$ vertices total, so the running time for the procedure is $O(n \log n)$ total. Finally, the boundary-to-boundary shortcut algorithm guarantees that for any pair of vertices u and v on A where the shortest path does not cross any cycle C of the cut open surface, we have $\mathrm{dist}_H(u,v) \le (1+\varepsilon)\mathrm{dist}_G(u,v)$. This proves the lemma since a shortest path that does cross a cycle C of the cut open surface is the concatenation of shortest paths that do not cross any such cycle. □

Lemma 7.5. *The total cost of all edges carrying shortcuts is $O(\varepsilon^{-6}\mathrm{OPT} \log W)$.*

Proof: By Lemma 4.3, the total cost of boundaries for regions without a hole is $O(\varepsilon^{-1}\mathrm{OPT} \log W)$. By Lemma 7.2, the total cost of all subgraphs A used in Lemma 7.4 for regions with holes is $O(\varepsilon^{-2}\mathrm{OPT} \log W)$. The current lemma follows from Lemmas 2.2 and 7.4. □

8 The spanner

In this section, we describe the final spanner construction for our algorithm and prove the construction follows the spanner properties. The spanner construction is summarized as Algorithm 2. The cheapest cycle other than the outer boundary enclosing a particular hole of a region R can be

Algorithm 2 Spanner(G)

```
1: $\mathcal{S} \leftarrow \emptyset$ ; $\mathcal{C} \leftarrow$ Skeleton($G, \lambda$)
2: add to $\mathcal{S}$ the edges of $\mathcal{C}$
3: for each region $R$ without a hole, add to $\mathcal{S}$ the edges carrying shortcuts in $R$
4: for each region $R$ with a hole, add to $\mathcal{S}$ the projection of the well-connected cover edges from $R^2$
5: for each region $R$ with a hole, add to $\mathcal{S}$ the projection of the edges carrying shortcuts in $R^2$
6: for each region $R$ with a hole, add to $\mathcal{S}$ the edges on the cheapest cycle in $R$ that encloses the
   heaviest hole of $R$ other than the outer boundary of $R$
```

computed in polynomial time using several instantiations of any polynomial time minimum s,t-cut algorithm [15].

To prove that our algorithm computes a spanner, we will iteratively replace cycles and their paths in a near-optimal solution with ones that lie in the spanner. The following lemmas will help us bound the total change in weight and cost from performing these operations.

Lemma 8.1. *Let O be a set of cycles. Let $K \in O$ be contained by region R, and let $r_G(K) > \varepsilon^{-1}\lambda$. Removing K from O changes the enclosed weight of O by at most $\varepsilon c(K)/\lambda$ and does not increase the cost of O.*

Lemma 8.2. *Let O be a set of cycles. Let $K \in O$ be strictly contained by region R, and let $r_G(K) \leq \varepsilon^{-1}\lambda$. Cycle K encloses the largest hole of R. Further, replacing K with the shortest cycle other than the outer boundary of R that encloses the largest hole of R changes the weight enclosed by O by at most $\varepsilon c(K)/\lambda$ and does not increase the cost of O.*

Proof: By Lemma 4.4, we know K must enclose the largest hole of R. Let A be the outer boundary of R and B be the largest hole of R. Let K' be the shortest cycle other than A enclosing B in R. Replacing K by a possibly cheaper cycle K' cannot increase the cost of O. Let $\Delta w(K)$ be the total weight of faces that move from being enclosed to not enclosed and vice versa after the replacement. Let $\tilde{\Delta} w(K)$ (respectively $\tilde{\Delta} w(K')$) be the total weight of faces that are enclosed by K (K') but not enclosed by B. Suppose $\tilde{\Delta} w(K) \geq \tilde{\Delta} w(K')$. By Lemma 4.4, we have $c(K)/\tilde{\Delta} w(K) > \varepsilon^{-1}\lambda$. In particular, $\Delta w(K) \leq \tilde{\Delta} w(K) < \varepsilon c(K)/\lambda$. If $\tilde{\Delta} w(K') > \tilde{\Delta} w(K)$, then either $\tilde{\Delta} w(K') = 0$ or K' is not in $\mathcal{C}$. Either way, we have $c(K')/\tilde{\Delta} w(K') > \varepsilon^{-1}\lambda$, and $\Delta w(K) \leq \tilde{\Delta} w(K') < \varepsilon c(K')/\lambda \leq \varepsilon c(K)/\lambda$. □

Lemma 8.3. *Let O be a set of cycles. Let $K \in O$ cross one or more cycles of the skeleton. Let p be a path of K's path decomposition lying in region R such that R has no holes and p has at least one edge disjoint from the spanner. Let u and v be the endpoints of p on the boundary of R. Finally, let p' be a shortcut in the spanner between u and v in R. Replacing p by p' changes the weight enclosed by O by at most $3\varepsilon c(p)/\lambda$ and increases the cost of O by at most $\varepsilon c(p)$.*

Proof: The bound on the cost change follows from the fact that shortcut p' is a $(1+\varepsilon)$-approximate shortest path. Let e be the edge of p strictly internal to R. Let $C = p \circ rev(p')$. Let C' be the cycle using darts of C that encloses and connects the boundary of faces of G enclosed by C. Cycle C' is strictly enclosed in R, because it contains e. Also, $c(C') \le c(C)$. The faces of G enclosed by C' are exactly the faces that switch sides when replacing p by p'. Therefore, the replacement will change the weight enclosed by O by exactly $w_G(C')$. By Lemma 4.4, we have $r_G(C') > \varepsilon^{-1}\lambda$. Recall, a shortcut is a $(1+\varepsilon)$-approximate shortest path. Therefore,

$$\varepsilon^{-1}\lambda < \frac{c(C')}{w_G(C')} = \frac{c(p)+c(p')}{w_G(C')} \le \frac{(2+\varepsilon)c(p)}{w_G(C')}.$$

In particular, $w_G(C') < \varepsilon(2+\varepsilon)c(p)/\lambda \le 3\varepsilon c(p)/\lambda$. □

Lemma 8.4. *Let O be a set of cycles. Let $K \in O$ cross one or more cycles of the skeleton. Let p be a path of K's path decomposition lying in region R such that R has at least one hole and p has at least one edge disjoint from the spanner. Let p' be a lift of p from R to R^2 with endpoints u and v. Finally, let p'' be a shortcut in the spanner between u and v in R^2. Replacing p by the projection of p'' changes the weight enclosed by O by at most $3\varepsilon c(p)/\lambda$ and increases the cost of O by at most $\varepsilon c(p)$.*

Proof: Again, the bound on cost change is immediate. Let C be the projection of $p' \circ \mathit{rev}(p'')$. By Lemma 6.2, C does not enclose the largest hole of R. The rest of the proof is identical to that of Lemma 8.3. □

We finally prove that our algorithm constructs a spanner.

Lemma 8.5. *The output of Algorithm 2 contains a solution with cost at most $(1+4\varepsilon)\mathrm{OPT}$ enclosing $b'W$ weight with $b' \in [b-6\varepsilon, b+6\varepsilon]$.*

Proof: Let O be the near-optimal solution given by
Lemma 7.3. We will modify O to create a solution O' that lies in the spanner.

Consider any cycle K in O. If it matches the hypothesis of Lemma 8.1, then it is simply removed from O. If it matches the hypothesis of Lemma 8.2, then it is replaced by the shortest cycle other than the outer boundary of its region enclosing the largest hole of its region. Line 6 of Algorithm 2 guarantees this cycle exists in the spanner.

In the final case, we know K has a path decomposition $p_0, p_1, \ldots$. We replace each p_i in the decomposition as follows. If p_i has no edges disjoint from the spanner, then it remains as is. If p_i has an edge strictly internal to a region without holes, then we replace p_i with a $(1+\varepsilon)$-approximate shortest path between its endpoints. Line 3 guarantees this path exists in the spanner. Finally, if p_i has an edge strictly internal to a region R with a hole, then p_i', p_i's lift to R^2, has both endpoints on the same component of R's well-connected cover graph. The output of the algorithm in Lemma 7.4 contains an $(1+\varepsilon)$-approximate shortest path p_i'' in the cyclic double cover between the endpoints of p_i'. Lines 4 and 5 of Algorithm 2 guarantee the projection of p_i'' exists in the spanner. Path p_i is replaced by the projection of p_i''. Any remaining cycles or decomposition paths of O exist in the skeleton itself, which is also in the spanner by Line 2 of Algorithm 2.

Solution O' is formed by performing the replacements described above. Let p be any cycle or path replaced above. By Lemmas 8.2, 8.3, and 8.4, after replacing p, the cost of O increases by at most $\varepsilon c(p)/\lambda$, and the weight of faces enclosed by O changes by at most $3\varepsilon c(p)/\lambda$. By summing over all cycles K, we see the total cost increases by at most $\varepsilon(1+2\varepsilon)\mathrm{OPT} \le 2\varepsilon\mathrm{OPT}$ (for $\varepsilon \le 1/2$) and the weight changes by at most $3\varepsilon(1+2\varepsilon)\mathrm{OPT}/\lambda \le 6\varepsilon W$. □

Lemma 8.6. *The output of Algorithm 2 has cost at most $O(\varepsilon^{-6}\mathrm{OPT}\log W)$.*

Proof: The shortest cycles enclosing largest holes within their region have total cost at most that of the skeleton itself. The rest of the spanner's components have their costs bounded in Lemmas 4.3, 7.2, and 7.5. □

Final remarks We have given a bicriteria approximation scheme, but there is much room for improvement. Is there an *efficient* PTAS, one whose running time is a polynomial with degree independent of ϵ? Is there a PTAS that does not approximate the balance b? Is there perhaps even a polynomial-time algorithm for bisection in planar graphs when the weights are small integers?

The research described in this paper began at a meeting at Brown University's Institute for Computational and Experimental Research in Mathematics. The authors thank Dániel Marx who described to the second author some results on fixed-parameter tractability of graph separation, leading to some discussion of the complexity in planar graphs and MohammadTaghi Hajiaghayi who asked whether there is an approximation scheme for bisection in planar graphs. Hajiaghayi had some initial thoughts due to its similarity to multiway cut for which there is a PTAS on planar graphs.

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