

# MINIMUM CYCLE AND HOMOLOGY BASES OF SURFACE-EMBEDDED GRAPHS\*

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ABSTRACT. We study the problems of finding a minimum cycle basis (a minimum-weight set of cycles that form a basis for the cycle space) and a minimum homology basis (a minimum-weight set of cycles that generates the 1-dimensional ( $\mathbb{Z}_2$ )-homology classes) of an undirected graph cellularly embedded on a surface. The problems are closely related, because the minimum cycle basis of a graph contains its minimum homology basis, and the minimum homology basis of the 1-skeleton of any graph is exactly its minimum cycle basis.

For the minimum cycle basis problem, we give a deterministic  $O(n^\omega + 2^{2g}n^2 + m)$ -time algorithm for graphs cellularly embedded on an orientable surface of genus  $g$ . Prior to this work, the best known algorithms for surface-embedded graphs were those for general graphs: an  $O(m^\omega)$ -time Monte Carlo algorithm [2] and a deterministic  $O(nm^2 / \log n + n^2m)$ -time algorithm [31].

For the minimum homology basis problem, we give a deterministic  $O((g+b)^3n \log n + m)$ -time algorithm for graphs cellularly embedded on an orientable or non-orientable surface of genus  $g$  with  $b$  boundary components, improving on existing algorithms for many values of  $g$  and  $n$ . The algorithm assumes that shortest paths are unique; this assumption can be avoided by either using random perturbations of the edge weights guaranteeing a high probability of success or by deterministic means at a cost of an  $O(\log n)$  factor increase in running time.

## 1 Introduction

### 1.1 Minimum cycle basis

Let  $G = (V, E)$  be a connected undirected graph with  $n$  vertices and  $m$  edges. We define a *cycle* of  $G$  to be a subset  $E' \subseteq E$  where each vertex  $v \in V$  is incident to an even number of edges in  $E'$ . The *cycle space* of  $G$  is the vector space over cycles in  $G$  where addition is defined as the symmetric difference of cycles' edge sets. It is well known that the cycle space of  $G$  is isomorphic to  $\mathbb{Z}_2^{m-n+1}$ ; in particular, the cycle space can be generated

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by the fundamental cycles of any spanning tree of  $G$ . A *cycle basis* is a maximal set of independent cycles. A *minimum cycle basis* is a cycle basis with a minimum number of edges (counted with multiplicity) or minimum total weight if edges are weighted<sup>1</sup>. Minimum cycle bases have applications in many areas such as electrical circuit theory [10,28], structural engineering [9], surface reconstruction [35], and the analysis of algorithms [30].

Sets of independent cycles form a matroid, so the minimum cycle basis can be computed using the standard greedy algorithm. However, there may be an exponential number of cycles in  $G$  from which to choose. Horton [25] gave the first polynomial time algorithm for the problem by observing that every cycle in the minimum cycle basis is the fundamental cycle of a shortest path tree, reducing the number of cycles to consider to  $O(nm)$ . Several other algorithms have been proposed to compute minimum cycle bases in general graphs [2,3,12,20,27,31]. The fastest of these algorithms are an  $O(m^\omega)$ -time Monte Carlo randomized algorithm of Amaldi *et al.* [2] and an  $O(nm^2/\log n + n^2m)$ -time deterministic algorithm of Mehlhorn and Michail [31]. Here,  $O(m^\omega)$  is the time it takes to multiply two  $m \times m$  matrices using fast matrix multiplication.

For the special case of *planar graphs*, faster algorithms are known. Hartvigsen and Mardon [23] observed that the cycles in the minimum cycle basis nest, and so can be represented by a tree; in fact, the edges of each cycle span an  $s, t$ -minimum cut between two vertices in the dual graph, and the Gomory-Hu tree [21] of the dual graph is precisely the tree of minimum cycle basis in the primal. Hartvigsen and Mardon [23] gave an  $O(n^2 \log n)$ -time algorithm for the minimum cycle basis problem in planar graphs, and Amaldi *et al.* [2] improved their running time to  $O(n^2)$ . Borradaile, Sankowski, and Wulff-Nilsen [4] showed how to compute an *oracle* for the minimum cycle basis and dual minimum cuts in  $O(n \log^4 n)$  time that is able to report individual cycles or cuts in time proportional to their size. Borradaile *et al.* [6] recently generalized the minimum cut oracle to graphs embeddable on an orientable surface of genus  $g$ . Their oracle takes  $2^{O(g^2)} n \log^3 n$  time to construct (improving upon the original planar oracle by a factor of  $\log n$ ). Unfortunately, their oracle does not help in finding the minimum cycle basis in higher genus graphs, because there is no longer a bijection between cuts in the dual graph and cycles in the primal graph.

That said, it is not surprising that the cycle basis oracle has not been generalized beyond the plane. While cuts in the dual continue to nest in higher genus surfaces, cycles do not. In fact, the minimum cycle basis of a toroidal graph must *always* contain at least one pair of crossing cycles, because any cycle basis must contain cycles which are topologically distinct. These cycles must represent different *homology classes* of the surface.

## 1.2 Minimum homology basis

Given a graph  $G$  embedded in a surface  $\Sigma$  of genus  $g$  with  $b$  boundary components, the homology of  $G$  is an algebraic description of the topology of  $\Sigma$  and of  $G$ 's embedding. In this paper, we focus on one-dimensional cellular homology over the finite field  $\mathbb{Z}_2$ . Homology of this type allows for simplified definitions. We say a cycle  $\eta$  is *null-homologous* if  $\eta$  is the

<sup>1</sup>There is a notion of minimum cycle bases in directed graphs as well, but we focus on the undirected case in this paper.

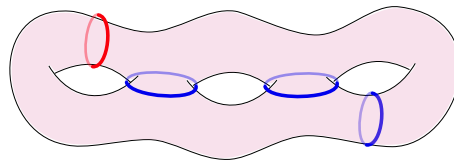


Figure 1. Two homologous cycles, one shown in red and the other in blue.

boundary of a subset of faces. Two cycles  $\eta$  and  $\eta'$  are *homologous* or in the same *homology class* if their symmetric difference  $\eta \oplus \eta'$  is null-homologous. Let  $\beta = g + \max\{b - 1, 0\}$  if  $\Sigma$  is *non-orientable* (contains a subset homeomorphic to the Möbius band), and let  $\beta = 2g + \max\{b - 1, 0\}$  otherwise. The homology classes form a vector space isomorphic to  $\mathbb{Z}_2^\beta$  known as the *homology space*. A *homology basis* of  $G$  is a set of  $\beta$  cycles in linearly independent homology classes, and the *minimum homology basis* of  $G$  is the homology basis with either the minimum number of edges or with minimum total weight if edges of  $G$  are weighted. Note that the cycle basis of an embedded graph is the homology basis of the same graph embedded on the same surface with the interior of all faces removed.

Erickson and Whittlesey [18] described an  $O(n^2 \log n + gn^2 + g^3n)$ -time algorithm for computing the minimum homology basis for orientable  $\Sigma$  without boundary. Like Horton [25], they apply the greedy matroid basis algorithm to a set of  $O(n^2)$  candidate cycles. Alternatively, a set of  $2^\beta$  candidate cycles containing the minimum homology basis can be computed easily by applying the algorithms of Italiano *et al.* [26] or Erickson and Nayyeri [17] for computing the minimum homologous cycle in any specified homology class. These algorithms take  $g^{O(g)}n \log \log n$  and  $2^{O(g)}n \log n$  time respectively. All three results mentioned above can be extended to surfaces with boundary (although Erickson and Whittlesey [18] do not explicitly state this). Similarly, the algorithms of Erickson and Whittlesey [18] and Erickson and Nayyeri [17] can compute the minimum homology basis for graphs embedded on non-orientable surfaces, although their paper only discusses orientable surfaces explicitly. Dey, Sun, and Wang [13] generalized the results above to arbitrary simplicial complexes, and Busaryev *et al.* [7] improved the running time of their generalization from  $O(n^4)$  to  $O(n^\omega + n^2g^{\omega-1})$ . Note that all of the algorithms above either take quadratic time in  $n$  (or worse) or they have exponential dependency on  $g$ . In contrast, it is well understood how to find a single cycle of the minimum homology basis of  $G$  in only  $O(g^2n \log n)$  time assuming orientable  $\Sigma$ , because the minimum-weight non-separating cycle will always be in the basis [8, 16].

### 1.3 Our results

We describe new algorithms for computing the minimum cycle basis and minimum homology basis of a cellularly surface-embedded graph  $G$ . Our algorithm for minimum cycle basis requires  $G$  be embedded on an orientable surface, but it is deterministic and runs in  $O(n^\omega + 2^{2g}n^2 + m)$  time, matching the running time of the randomized algorithm of Amaldi *et al.* [2] when  $g$  is sufficiently small. Our algorithm for minimum homology basis works for orientable or non-orientable surfaces, is also deterministic, and it runs in  $O((g + b)^3n \log n + m)$  time, assuming shortest paths are unique. The assumption of unique

shortest paths is only necessary to use the multiple-source shortest path data structure of Cabello, Chambers, and Erickson [8]. This assumption can be avoided by either using random perturbations of the edge weights guaranteeing a high probability of success or by deterministic means at a cost of an  $O(\log n)$  factor increase in running time [8]. For simplicity, we will assume shortest paths are unique during the presentation of our minimum homology basis algorithm. In any case, this is the first algorithm for minimum homology basis that has a running time simultaneously near-linear in  $n$  and polynomial in  $g$ .

At a high level, both of our algorithms are based on the  $O(nm^2 + n^2m \log n)$ -time algorithm of Kavitha *et al.* [27] who in turn use an idea of de Pina [12]. We compute our basis cycles one by one. Over the course of the algorithm, we maintain a set of *support vectors* that form the basis of the subspace that is *orthogonal* to the set of cycles we have already computed. Every time we compute a new cycle, we find one of minimum weight that is *not* orthogonal to a chosen support vector  $S$ , and then update the remaining support vectors so they remain orthogonal to our now larger set of cycles. Using the divide-and-conquer approach of Kavitha *et al.* [27], we are able to maintain these support vectors in only  $O(n^\omega)$  time total in our minimum cycle basis algorithm and  $O(g^\omega)$  time total in our minimum homology basis algorithm. Our approaches for picking the minimum-weight cycle not orthogonal to  $S$  form the more technically interesting parts of our algorithms and are unique to this work.

For our minimum cycle basis algorithm, we compute a collection of  $O(2^{2g}n)$  cycles that contain the minimum cycle basis and then partition these cycles according to their homology classes. The cycles within a single homology class nest in a similar fashion to the minimum cycle basis cycles of a planar graph. Every time we compute a new cycle for our minimum cycle basis, we walk up the  $2^{2g}$  trees of nested cycles and find the minimum-weight cycle not orthogonal to  $S$  in  $O(n)$  time per tree. Overall, we spend  $O(2^{2g}n^2)$  time finding these cycles; if any improvement is made on the time it takes to update the support vectors, then the running time of our algorithm as a whole will improve as well.

Our minimum homology basis algorithm uses a covering space called the cyclic double cover. As shown by Erickson [16], the cyclic double cover provides a convenient way to find a minimum-weight closed walk  $\gamma$  crossing an arbitrary non-separating cycle  $\lambda$  an odd number of times. We extend his construction so that we may consider not just one  $\lambda$  but any arbitrarily large collection of cycles. Every time we compute a new cycle in our minimum homology basis algorithm, we let  $S$  determine a set of cycles that must be crossed an odd number of times, build the cyclic double cover for that set, and then compute our homology basis cycle in  $O((g+b)gn \log n)$  time by computing minimum-weight paths in the covering space<sup>2</sup>.

The rest of the paper is organized as follows. We provide more preliminary material on surface-embedded graphs in Section 2. In Section 3, we describe a characterization of cycles and homology classes using binary vectors. These vectors are helpful in formally

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<sup>2</sup>In addition to the above results, we note that it is possible to improve the  $g^{O(g)}n \log \log n$ -time algorithm for minimum homology basis based on Italiano *et al.* [26] so that it runs in  $2^{O(g)}n \log \log n$  time. However, this improvement is a trivial adaption of techniques used by Fox [19] to get a  $2^{O(g)}n \log \log n$ -time algorithm for minimum-weight non-separating and non-contractible cycle in undirected graphs. We will not further discuss this improvement in our paper.

defining our support vectors. We give a high level overview of our minimum cycle basis algorithm in Section 4 and describe how to pick individual cycles in Section 5. Finally, we give our minimum homology basis algorithm in Section 6.

## 2 Preliminaries

We begin with an overview of graph embeddings on surfaces. For more background, we refer readers to books and surveys on topology [24,33], computational topology [14,37], and graphs on surfaces [11,32].

A *surface* or 2-manifold with boundary  $\Sigma$  is a compact Hausdorff space in which every point lies in an open neighborhood homeomorphic to the Euclidean plane or the closed half plane. The *boundary* of the surface is the set of all points whose open neighborhoods are homeomorphic to the closed half plane. Every boundary component is homeomorphic to the circle. A *cycle* in the surface  $\Sigma$  is a continuous function  $\gamma : S^1 \rightarrow \Sigma$ , where  $S^1$  is the unit circle. A cycle  $\gamma$  is called *simple* if  $\gamma$  is injective. A *path*  $p$  in surface  $\Sigma$  is a continuous function  $p : [0, 1] \rightarrow \Sigma$ ; again,  $p$  is simple if  $p$  is injective. A *loop* is a path  $p$  such that  $p(0) = p(1)$ ; equivalently, it is a cycle with a designated basepoint. The *genus* of the surface  $\Sigma$ , denoted by  $g$ , is the maximum number of disjoint simple cycles  $\gamma_1, \dots, \gamma_g$  in  $\Sigma$  such that  $\Sigma \setminus (\gamma_1 \cup \dots \cup \gamma_g)$  is connected. A surface  $\Sigma$  is *non-orientable* if it contains a subset homeomorphic to the Möbius band; otherwise, it is *orientable*. A surface is characterized up to homeomorphism by its genus, number of boundary components, and whether or not it is orientable.

The *embedding* of graph  $G = (V, E)$  is a drawing of  $G$  on  $\Sigma$  which maps vertices to distinct points on  $\Sigma$  and edges to internally disjoint simple paths whose endpoints lie on their incident vertices' points. A *face* of the embedding is a maximally connected subset of  $\Sigma$  that does not intersect the image of  $G$ . An embedding is *cellular* if every face is homeomorphic to an open disc; in particular, every boundary component must be covered by (the image of) a cycle in  $G$ . These boundary cycles must be vertex-disjoint. We consider only cellular embeddings in this paper. Such embeddings can be described combinatorially using a rotation system and a signature. The *rotation system* describes the cyclic ordering of edges around each vertex. The *orientation signature*  $\text{sig} : E \rightarrow \{0, 1\}$  is a function that assigns to each edge  $e$  a bit. Value  $\text{sig}(e) = 0$  if the cyclic ordering of  $e$ 's endpoints are in the same direction; otherwise,  $\text{sig}(e) = 1$ . Abusing notation, we denote the orientation signature of a cycle  $\eta$  (in  $G$ ) as  $\text{sig}(\eta)$  and define it as the exclusive-or of its edges' orientation signatures. If  $\text{sig}(\eta) = 1$ , we say  $\eta$  is *one-sided*. Otherwise, we say that  $\eta$  is *two-sided*. Surface  $\Sigma$  is orientable if and only if every cycle of  $G$  is two-sided.

Let  $F$  be the set of faces in  $G$ . Let  $n$ ,  $m$ ,  $\ell$ , and  $b$  be the number of vertices, edges, faces, and boundary components of  $G$ 's embedding respectively. The *Euler characteristic*  $\chi$  of  $\Sigma$  is  $2 - 2g - b$  if  $\Sigma$  is orientable and is  $2 - g - b$  otherwise. By Euler's formula,  $\chi = n - m + \ell$ . Embedded graphs can be *dualized*:  $G^*$  is a graph embedded on the surface obtained by attaching disks to every boundary component of  $\Sigma$ . There is a vertex in  $G^*$  for every face and boundary component in  $G$  and a face in  $G^*$  for every vertex of  $G$ . We refer to the dual vertices of boundary components as *boundary dual vertices*. Two vertices in

$G^*$  are adjacent if the corresponding faces/boundary components are separated by an edge in  $G$ . We generally do not distinguish between edges in the primal and dual graphs. We assume  $\Sigma$  contains at least one boundary component as it does not affect the homology of  $\Sigma$  to remove a face when there is no boundary to begin with. In particular, this assumption simplifies the definition of  $\beta$  as given in the introduction so  $\beta = g + b - 1$  if  $\Sigma$  is non-orientable and  $\beta = 2g + b - 1$  otherwise.

A **spanning tree** of the graph  $G$  is a subset of edges of  $G$  which form a tree containing every vertex. A **spanning coforest** is a subset of edges which form a forest in the dual graph with exactly  $b$  components, each containing one dual boundary vertex. A **tree-coforest decomposition** of  $G$  is a partition of  $G$  into 3 edge disjoint subsets,  $(T, L, C)$ , where  $T$  is a spanning tree of  $G$ ,  $C$  is a spanning coforest, and  $L$  is the set of leftover edges  $E \setminus (T \cup C)$  [15, 17]. Euler's formula implies  $|L| = \beta$ .

A  **$w, w'$ -path**  $p$  (in  $G$ ) is an ordered sequence of edges  $\{u_1v_1, u_2v_2, \dots, u_kv_k\}$  where  $w = u_1$ ,  $w' = v_k$ , and  $v_i = u_{i+1}$  for all positive  $i < k$ ; a **closed path** is a path which starts and ends on the same vertex. A path is **simple** if it repeats no vertices (except possibly the first and last). We sometimes use **simple cycle** to mean a simple closed path. A path in the dual graph  $G^*$  is referred to as a **co-path** and a cycle in  $G^*$  is referred to as a **co-cycle**. Simple paths and cycles in the dual are referred to as simple co-paths and co-cycles respectively. Every member of the minimum cycle basis (and subsequently the minimum homology basis) is a simple cycle [25]. We let  $\sigma(u, v)$  denote an arbitrary shortest (minimum-weight)  $u, v$ -path in  $G$ . Let  $p[u, v]$  denote the **subpath** of  $p$  from  $u$  to  $v$ . Given a  $u, v$ -path  $p$  and a  $v, w$ -path  $p'$ , let  $p \cdot p'$  denote their concatenation. Two paths  $p$  and  $p'$  **cross** if their embeddings in  $\Sigma$  cannot be made disjoint through infinitesimal perturbations; more formally, they cross if there is a maximal (possibly trivial) common subpath  $p''$  of  $p$  and  $p'$  such that, upon contracting  $p''$  to a vertex  $v$ , two edges each of  $p$  and  $p'$  alternate in their embedded around  $v$ . Two closed paths cross if they have subpaths which cross.

Let  $\gamma$  be a closed path in  $G$  that does not cross itself. We define the operation of **cutting** along  $\gamma$  and denote it  $G \not\sim \gamma$ . The graph  $G \not\sim \gamma$  is obtained by cutting along  $\gamma$  in the drawing of  $G$  on the surface, creating two copies of  $\gamma$ . If  $\text{sig}(\gamma) = 0$ , then the two copies of  $\gamma$  each form boundary components in the cut open surface. Otherwise, the two copies of  $\gamma$  together form a single closed path that is the concatenation of  $\gamma$  to itself at both ends; the single closed path forms a single boundary component. Likewise, given a simple path  $\sigma$  in  $G$ , we obtain the graph  $G \not\sim \sigma$  by cutting along  $\sigma$ , creating two interiorly disjoint copies connected at their endpoints. The cut open surface has one new boundary component bounded by the copies of  $\sigma$ .

Let  $F'$  be a collection of faces and boundary components. Let  $\partial F'$  denote the boundary of  $F'$ , the set of edges with exactly one incident face or boundary component in  $F'$ . We sometimes call  $F'$  a **cut** of  $G^*$  and say  $\partial F'$  **spans** the cut. A co-path  $p$  with edge  $uv \in \partial F'$  **crosses** the cut at  $uv$ .

Finally, let  $w$  and  $w'$  be two bit-vectors of the same length. We let  $\langle w, w' \rangle$  denote the **dot product** of  $w$  and  $w'$ , defined by the exclusive-or of the products of their corresponding bits.

## 2.1 Sparsifying $G$

We assume  $g = O(n^{1-\varepsilon})$  for some constant  $\varepsilon > 0$ ; otherwise, our algorithms offer no improvement over previously known results. Because boundary cycles are vertex-disjoint, we also have  $b = O(n)$ . We show below that we can assume without loss of generality that  $G$  has  $O(n)$  edges and faces. Essentially, our technique is to remove faces of degree 1 or 2 without increasing the number of vertices; Euler's formula immediately limits the number of edges and faces in the modified graph to  $O(n)$ . To that end, we need to remove parallel edges and loops from  $G$ .

For the minimum cycle basis, we begin by computing all pairs of shortest paths in  $O(n^2 \log n + m)$  time by ignoring all but the lightest edge in each set of parallel edges, allowing Dijkstra's algorithm to run in  $O(n \log n)$  time per instantiation for  $n$  instantiations. This computation is not required for the homology basis problem. We iteratively perform the following procedure until every face has degree 3 or greater or our graph is one of a constant number of easy cases. In each iteration, we add at most one cycle to the minimum cycle basis or minimum homology basis. Let  $f$  be a face of degree 1 or 2. If  $f$  has degree 1, then it is bounded by a null-homologous loop  $e$  in  $G$ . We add  $\{e\}$  to the minimum cycle basis, because it is the cheapest cycle containing  $e$ , but we do not add it to the minimum homology basis. If  $G$  consists entirely of  $e$ , we terminate; otherwise we remove  $e$  and  $f$  from the graph and continue with the next iteration. If  $f$  has degree 2, then it is either bounded by two distinct edges  $e$  and  $e'$  or bounded twice by a single edge. In the latter case, graph  $G$  must be the path of length 1 embedded in the sphere or it is a single vertex and non-null homologous loop embedded in the projective plane (the non-orientable surface of genus 1). If it is the path in the sphere, we add nothing to the minimum cycle basis and minimum homology basis, and we terminate. If it is a loop in the projective plane, we add it to both the minimum cycle basis and the minimum homology basis and terminate. Now suppose  $f$  is bounded by distinct faces  $e$  and  $e'$ , and let  $e$  have less weight than  $e'$  without loss of generality. Edge  $e'$  belongs to cycle  $\{e, e'\}$ , so it belongs to some cycle of the minimum cycle basis. Let  $\sigma$  be the shortest path between the endpoints of  $e'$ . We add  $\sigma \cdot e'$  to the minimum cycle basis. No other cycle in the minimum cycle basis contains  $e'$ , because it would always be at least as cheap to include  $e$  in the cycle instead. Also, no cycle of the minimum homology basis contains  $e'$ , because it would always be at least as cheap to include  $e$  and  $\{e, e'\}$  itself is null-homologous. We remove  $e'$  and  $f$  from the graph and continue with the next iteration.

Each iteration is done in constant time, and there are at most  $m$  iterations of the above algorithm. Therefore, the preprocessing procedure takes  $O(n^2 \log n + m)$  time total. We assume for the rest of that paper that  $m = O(n)$  and  $\ell = O(n)$ .

## 3 Cycle and Homology Signatures

We begin the presentation of our algorithms by giving a characterization of cycles and homology classes using binary vectors. These vectors will be useful in helping us determine which cycles can be safely added to our minimum cycle and homology bases. Let  $(T, L, C)$  be an arbitrary tree-coforest decomposition of  $G$ ; set  $L$  contains exactly  $\beta$  edges  $e_1, \dots, e_\beta$ .

For each index  $i \in \{1, \dots, \beta\}$ , graph  $C \cup \{e_i\}$  contains a unique simple co-cycle or a unique simple co-path between distinct dual boundary vertices. Let  $p_i$  denote this simple co-cycle or co-path. Let  $f_{\beta+1}, \dots, f_{m-n+1}$  denote the  $m-n+1-\beta = \ell$  faces of  $G$ , and for each index  $i \in \{\beta+1, \dots, m-n+1\}$ , let  $p_i$  denote the simple co-path from  $f_i$  to the dual boundary vertex in  $f_i$ 's component of  $C$ .

For each edge  $e$  in  $G$ , we define its **cycle signature**  $[e]$  as an  $(m-n+1)$ -bit vector whose  $i$ th bit is equal to 1 if and only if  $e$  appears in  $p_i$ . The cycle signature  $[\eta]$  of any cycle  $\eta$  is the bitwise exclusive-or of the signatures of its edges. Equivalently, the  $i$ th bit of  $[\eta]$  is 1 if and only if  $\eta$  and  $p_i$  share an odd number of edges. Similarly, for each edge  $e$  in  $G$ , we define its **homology signature**  $[e]_h$  as a  $\beta$ -bit vector whose  $i$ th bit is equal to 1 if and only if  $e$  appears in  $p_i$ . The homology signature of cycles is defined similarly.

The following lemma is immediate.

**Lemma 3.1.** *Let  $\eta$  and  $\eta'$  be two cycles. We have  $[\eta \oplus \eta'] = [\eta] \oplus [\eta']$  and  $[\eta \oplus \eta']_h = [\eta]_h \oplus [\eta']_h$ .*

Let  $\zeta_i$  denote the unique simple cycle in  $T \cup \{e_i\}$ . The following lemma helps us explain the properties of cycle and homology signatures.

**Lemma 3.2.** *The set of cycles  $\{\zeta_1, \dots, \zeta_\beta\}$  form a homology basis.*

**Proof:** We prove that the cycles lie in independent homology classes by showing that the symmetric difference of any non-empty subset of  $\{\zeta_1, \dots, \zeta_\beta\}$  is not null-homologous. Suppose to the contrary that there exists a non-empty  $\Upsilon \subseteq \{\zeta_1, \dots, \zeta_\beta\}$  such that  $\bigoplus_{\eta \in \Upsilon} \eta = \partial F'$  for some subset of faces  $F' \subseteq F$ , where  $\bigoplus$  is the symmetric difference of its operands. Let  $\zeta_i \in \Upsilon$  be an arbitrary member of the subset. Co-path  $p_i$  shares exactly one edge with  $\zeta_i$ , and it shares no edges with any other  $\eta \in \Upsilon$ . In particular,  $p_i$  crosses dual cut  $F'$  an odd number of times. Therefore,  $p_i$  cannot be a co-cycle. Further,  $p_i$  cannot be a co-path between two distinct dual boundary vertices, because exactly one of those two vertices would have to lie inside  $F'$ , a contradiction on  $F'$  only containing faces. We conclude  $\Upsilon$  cannot exist and the cycles  $\{\zeta_1, \dots, \zeta_\beta\}$  do lie in independent homology classes.  $\square$

Let  $w$  be an arbitrary  $(m-n+1)$ -bit vector. We construct a cycle  $\eta_w$  to demonstrate how cycle and homology signatures provide a convenient way to distinguish between cycles and their homology classes. Let  $\Upsilon \subseteq \{\zeta_1, \dots, \zeta_\beta\}$  be the subset of basis cycles containing exactly the cycles  $\zeta_i$  such that the  $i$ th bit of  $w$  is equal to 1. Similarly, let  $F' \subseteq F$  be the subset of faces such that face  $f_i \in F'$  if and only the  $i$ th bit of  $w$  is equal to 1. Let  $\eta_w = \bigoplus_{\eta \in (\Upsilon \cup \{\partial F'\})} \eta$ .

**Lemma 3.3.** *We have  $[\eta_w] = w$ .*

**Proof:** Let  $i \in \{1, \dots, m-n+1\}$ , and let  $p_i$  be the co-path as defined above. Suppose  $i \in \{1, \dots, \beta\}$ . Co-path  $p_i$  crosses cut  $F'$  an even number of times. If bit  $i$  in  $w$  is set to 1, then  $p_i$  shares exactly one edge of  $\bigoplus_{\eta \in \Upsilon} \eta$  by construction, and it must share an odd number of



edges with  $\eta_w$  as well. If bit  $i$  in  $w$  is set to 0, then  $p_i$  shares no edges with  $\bigoplus_{\eta \in \Upsilon} \eta$ , and it must share an even number of edges with  $\eta_w$ .

Now, suppose  $i \in \{\beta + 1, \dots, m - n + 1\}$ . Co-path  $p_i$  shares no edges with  $\bigoplus_{\eta \in \Upsilon} \eta$ . If  $i$  is set to 1, then  $f_i \in F'$  and  $p_i$  crosses cut  $F'$  an odd number of times. Therefore, it shares an odd number of edges with  $\eta_w$ . If  $i$  is set to 0, then  $f_i \notin F'$ , and  $p_i$  crosses cut  $F'$  an even number of times, sharing an even number of edges with  $\eta_w$ .  $\square$

**Corollary 3.4.** *Let  $\eta$  and  $\eta'$  be two cycles. We have  $\eta = \eta'$  if and only if  $[\eta] = [\eta']$ .*

Observe that the homology class of  $\eta_w$  is entirely determined by the first  $\beta$  bits of  $w$ . We immediately obtain an alternative (and more combinatorially inspired) proof of the following corollary of Erickson and Nayyeri [17].

**Corollary 3.5 (Erickson and Nayyeri [17, Corollary 3.3]).** *Two cycles  $\eta$  and  $\eta'$  are homologous if and only if  $[\eta]_h = [\eta']_h$ .*

**Corollary 3.6.** *Cycle signatures are an isomorphism between the cycle space and  $\mathbb{Z}_2^{m-n+1}$ , and homology signatures are an isomorphism between the first homology space and  $\mathbb{Z}_2^{2g}$ .*

## 4 Minimum Cycle Basis

We now describe our algorithm for computing a minimum cycle basis. We assume without loss of generality that surface  $\Sigma$  contains exactly one boundary component, because the addition or removal of boundary does not affect the cycles of  $G$ . We denote the one boundary component and its corresponding dual boundary vertex as  $f_\infty$ , since we will use it in our algorithm in a fashion analogous to the infinite face of a planar graph. Our algorithm for minimum cycle basis is only for a graph  $G$  embedded on an orientable surface  $\Sigma$ . We conclude  $\beta = 2g$ .

Our algorithm is based on one of Kavitha, Mehlhorn, Michail and Paluch [27] which is in turn based on an algorithm of de Pina [12]. Our algorithm incrementally adds simple cycles  $\gamma_1, \dots, \gamma_{m-n+1}$  to the minimum cycle basis. In order to do so, it maintains a set of  $(m - n + 1)$ -bit **support vectors**  $S_1, \dots, S_{m-n+1}$  with the following properties:

- The support vectors form a basis for  $\mathbb{Z}_2^{m-n+1}$ .
- When the algorithm is about to compute the  $j$ th simple cycle  $\gamma_j$  for the minimum cycle basis,  $\langle S_j, [\gamma_{j'}] \rangle = 0$  for all  $j' < j$ .

Our algorithm chooses for each cycle  $\gamma_j$  the minimum-weight cycle  $\gamma$  such that  $\langle S_j, [\gamma] \rangle = 1$ . Note that  $S_j$  must have at least one bit set to 1, because the set of vectors  $S_1, \dots, S_{m-n+1}$  forms a basis. Therefore, such a  $\gamma$  does exist; in particular, we could choose  $[\gamma]$  to contain exactly one bit equal to 1 which matches any 1-bit of  $S_j$ . The second property ensures that the rank of  $\{\gamma_1, \dots, \gamma_j\}$  is  $j$ ; in particular,  $\{\gamma_1, \dots, \gamma_{m-n+1}\}$  has rank  $m - n + 1$  and is therefore a basis. The correctness of choosing  $\gamma_j$  as above is guaranteed by the following lemma.

**Lemma 4.1.** *Let  $S$  be an  $(m - n + 1)$ -bit vector with at least one bit set to 1, and let  $\eta$  be the minimum-weight cycle such that  $\langle S, [\eta] \rangle = 1$ . Then,  $\eta$  is a member of the minimum cycle basis.*

**Proof:** Let  $\eta_1, \dots, \eta_{2^{m-n+1}}$  be the collection of cycles ordered by increasing weight, and choose  $j$  such that  $\eta_j = \eta$ . For any subset  $\Upsilon$  of  $\{\eta_1, \dots, \eta_{j-1}\}$ , we have  $\langle [\bigoplus_{\eta' \in \Upsilon} \eta'], S \rangle = 0$ . Therefore,  $\eta$  is independent of  $\{\eta_1, \dots, \eta_{j-1}\}$ . It is well-known that sets of independent cycles form a matroid. Further, the greedy algorithm of ordering a matroids' elements by weight and iteratively growing an independent set by adding the minimum-weight element keeping the set independent is optimal for finding a minimum basis. Therefore,  $\eta$  is a member of the minimum cycle basis.  $\square$

Our algorithm updates the support vectors and computes minimum cycle basis vectors in a recursive manner. Initially, each support vector  $S_i$  has only its  $i$ th bit set to 1. Borrowing nomenclature from Kavitha *et al.* [27], we define two procedures,  $\text{extend}(j, k)$  which extends the current set of basis cycles by adding  $k$  cycles starting with  $\gamma_j$ , and  $\text{update}(j, k)$  which updates support vectors  $S_{j+\lfloor k/2 \rfloor}, \dots, S_{j+k-1}$  so that for any  $j', j''$  with  $j + \lfloor k/2 \rfloor \leq j' < j + k$  and  $1 \leq j'' < j + \lfloor k/2 \rfloor$ , we have  $\langle S_{j'}, [\gamma_{j''}] \rangle = 0$ . Our algorithm runs  $\text{extend}(1, m - n + 1)$  to compute the minimum cycle basis.

We implement  $\text{extend}(j, k)$  in the following manner: If  $k > 1$ , then our algorithm recursively calls  $\text{extend}(j, \lfloor k/2 \rfloor)$  to add  $\lfloor k/2 \rfloor$  cycles to the partial minimum cycle basis. It then calls  $\text{update}(j, k)$  so that support vectors  $S_{j+\lfloor k/2 \rfloor}, \dots, S_{j+k-1}$  become orthogonal to the newly added cycles of the partial basis. Finally, it computes the remaining  $\lfloor k/2 \rfloor$  basis cycles by calling  $\text{extend}(j + \lfloor k/2 \rfloor, \lfloor k/2 \rfloor)$ . If  $k = 1$ , then  $\langle S_j, [\gamma_{j'}] \rangle = 0$  for all  $j' < j$ . Our algorithm is ready to find basis cycle  $\gamma_j$ . We describe an  $O(2^{2g}n)$ -time procedure to find  $\gamma_j$  in Section 5.

We now describe  $\text{update}(j, k)$  in more detail. Our algorithm updates each support vector  $S_{j'}$  where  $j + \lfloor k/2 \rfloor \leq j' < j + k$ . The vector  $S_{j'}$  becomes  $S'_{j'} = S_{j'} + \alpha_{j',0}S_j + \alpha_{j',1}S_{j+1} + \dots + \alpha_{j',(\lfloor k/2 \rfloor - 1)}S_{j+\lfloor k/2 \rfloor - 1}$  for some set of scalar bits  $\alpha_{j',0} \dots \alpha_{j',(\lfloor k/2 \rfloor - 1)}$ . After updating, the set of vectors  $S_1, \dots, S_{m-n+1}$  remains a basis for  $\mathbb{Z}_2^{m-n+1}$  regardless of the choices for the  $\alpha$  bits. Note that  $\text{extend}(j, k)$  is only called after support vectors  $S_j, \dots, S_{j+k-1}$  are updated to be orthogonal to each minimum basis cycle  $\gamma_{j''}$  with  $j'' < j$ . Therefore, every linear combination of support vectors  $S_j, \dots, S_{j+\lfloor k/2 \rfloor}$  is orthogonal to each  $\gamma_{j''}$  with  $j'' < j$ . In turn, we see  $\langle S'_{j'}, [\gamma_{j''}] \rangle = 0$  for all  $j'' < j$  for all choices of the  $\alpha$  bits.

However, it is non-trivial to guarantee  $\langle S'_{j'}, [\gamma_{j''}] \rangle = 0$  for all  $j''$  where  $j \leq j'' < j + \lfloor k/2 \rfloor$ . Let  $w^T$  denote the transpose of a vector  $w$ . Let

$$X = \begin{pmatrix} S_j \\ \dots \\ S_{j+\lfloor k/2 \rfloor - 1} \end{pmatrix} \cdot ([\gamma_j]^T \dots [\gamma_{j+\lfloor k/2 \rfloor - 1}]^T)$$

and

$$Y = \begin{pmatrix} S_{j+\lfloor k/2 \rfloor} \\ \dots \\ S_{j+k-1} \end{pmatrix} \cdot ([\gamma_j]^T \dots [\gamma_{j+\lfloor k/2 \rfloor - 1}]^T).$$

Let  $A = YX^{-1}$ . Row  $j' - j - \lfloor k/2 \rfloor + 1$  of matrix  $A$  contains exactly the bits  $\alpha_{j',0} \dots \alpha_{j',(\lfloor k/2 \rfloor - 1)}$  we are seeking [27, Section 4]. Matrices  $X$ ,  $Y$ , and  $A$  can be computed in  $O(nk^{\omega-1})$  time using fast matrix multiplication and inversion, implying that the new support vectors  $S'_{j+\lfloor k/2 \rfloor}, \dots, S'_{j+k-1}$  can be computed in the same amount of time.

We can bound the running time of  $\text{extend}(j, k)$  using the following recurrence:

$$T(k) = \begin{cases} 2T(k/2) + O(nk^{\omega-1}) & \text{if } k > 1 \\ O(2^{2g}n) & \text{if } k = 1 \end{cases}$$

The total time spent in calls to  $\text{extend}(j, k)$  where  $k > 1$  is  $O(nk^{\omega-1})$ , assuming  $\omega > 2$ . The total time spent in calls to  $\text{extend}(j, 1)$  is  $O(2^{2g}nk)$ . Therefore,  $T(k) = O(nk^{\omega-1} + 2^{2g}nk)$ . The running time of our minimum cycle basis algorithm (after sparsifying  $G$ ) is  $T(O(n)) = O(n^{\omega} + 2^{2g}n^2)$ .

## 5 Selecting Cycles

A *Horton cycle* is a simple cycle given by a shortest  $x, u$ -path, a shortest  $x, v$ -path, and the edge  $uv$ ; in particular, the set of all Horton cycles is given by the set of  $m - n + 1$  elementary cycles for each of the  $n$  shortest path trees [25]. Thus, in sparse graphs, there are  $O(n^2)$  Horton cycles. A simple cycle  $\gamma$  of a graph  $G$  is *isometric* if for every pair of vertices  $x, y \in \gamma$ ,  $\gamma$  contains a shortest  $x, y$ -path. Hartvigsen and Mardon prove that the cycles of any minimum cycle basis are all isometric [23]. Therefore, it suffices for us to focus on the set of isometric cycles to find the cycle  $\gamma_j$  as needed for Section 4. Amaldi *et al.* [2] show how to extract the set of distinct isometric cycles from a set of Horton cycles in  $O(nm)$  time. Each isometric cycle is identified by a shortest path tree's root and a non-tree edge.

Here, we show that there are at most  $O(2^{2g}n)$  isometric cycles in our graph of genus  $g$  (Section 5.1), and they can be partitioned into sets according to their homology classes. We can represent the isometric cycles in a given homology class using a tree that can be built in  $O(n^2)$  time (Section 5.2). We then show that we can use these trees to find the minimum-cost cycle  $\gamma_j$  as needed for Section 4 in linear time per homology class of isometric cycles. We close with a discussion on how to improve the running time for computing and representing isometric cycles (Section 5.4). We note that these improvements do not improve the overall running time of our algorithm, since by maintaining separate representations of the cycles according to their homology class, we require linear time per representation to process the support vector with respect to which  $\gamma_j$  is non-orthogonal; we also require  $O(n^{\omega})$  time to update the support vectors. However, they do further emphasize the bottleneck our algorithm faces in updating and representing the support vectors.

### 5.1 Isometric cycles in orientable surfaces

Here we prove some additional structural properties that isometric cycles have in orientable surface-embedded graphs. To this end, we herein assume that shortest paths are unique.

Hartvigsen and Mardon show how to achieve this assumption algorithmically when, in particular, all pairs of shortest paths are computed, as we do [23]. We first prove a generalization of the following lemma for the planar case by Borradaile, Sankowski and Wulff-Nilsen.

**Lemma 5.1 (Borradaile *et al.* [4, Lemma 1.4]).** *Let  $G$  be a graph in which shortest paths are unique. The intersection between an isometric cycle and a shortest path in  $G$  is a (possibly empty) shortest path. The intersection between two distinct isometric cycles  $\gamma$  and  $\gamma'$  in  $G$  is a (possibly empty) shortest path; in particular, if  $G$  is a planar embedded graph,  $\gamma$  and  $\gamma'$  do not cross.*

**Lemma 5.2.** *Two isometric cycles in a given homology class in a graph with unique shortest paths do not cross.*

**Proof:** Let  $\gamma$  and  $\gamma'$  be two isometric cycles in a given homology class. Suppose for a contradiction that  $\gamma$  and  $\gamma'$  cross. By the second part of Lemma 5.1, and the assumption that  $\gamma$  and  $\gamma'$  cross,  $\gamma \cap \gamma'$  is a single simple path  $p$ . Therefore,  $\gamma$  and  $\gamma'$  cross exactly once.

Suppose  $\gamma$  and  $\gamma'$  are not null-homologous. Cutting the surface open along  $\gamma$  results in a connected surface with two boundary components which are connected by  $\gamma'$ . Cutting the surface further along  $\gamma'$  does not disconnect the surface. Therefore  $\gamma \oplus \gamma'$  does not disconnect the surface, and so  $\gamma$  and  $\gamma'$  are not homologous, a contradiction.

If  $\gamma$  and  $\gamma'$  are null-homologous, then cutting the surface open along  $\gamma$  results in a disconnected surface in which  $\gamma' \setminus p$  is a path, but between different components of the surface, a contradiction.  $\square$

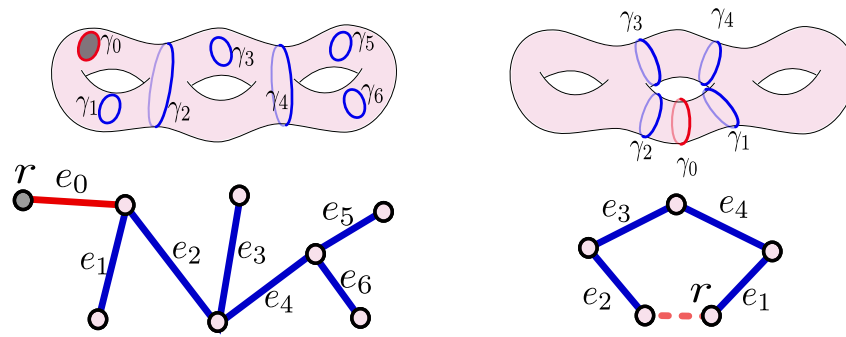
**Corollary 5.3.** *There are at most  $\ell$  distinct isometric cycles in a given homology class in a graph with  $\ell$  faces and unique shortest paths.*

**Proof:** Consider the set  $\{C_1, C_2, \dots\}$  of distinct isometric cycles in a given homology class other than the null homology class. We prove by induction that  $\{C_1, C_2, \dots, C_i\}$  cut the surface into non-trivial components, each of which is bounded by exactly two of  $C_1, C_2, \dots, C_i$ ; this is true for  $C_1, C_2$  since they are homologous, distinct and do not cross.  $C_{i+1}$  must be contained in one component, bounded by, say,  $C_j$  and  $C_k$  since  $C_{i+1}$  does not cross any other cycle. Cutting this component along  $C_{i+1}$  creates two components bounded by  $C_j, C_{i+1}$  and  $C_k, C_{i+1}$ , respectively. Since the cycles are distinct, these component must each contain at least one face. A similar argument holds for the set of null-homologous isometric cycles.  $\square$

Since there are  $2^{2g}$  homology classes and  $\ell = O(n)$ , we get:

**Corollary 5.4.** *There are  $O(2^{2g}n)$  distinct isometric cycles in a graph of orientable genus  $g$  with unique shortest paths.*

We remark that Lemma 5.2 is not true for graphs embedded in non-orientable surfaces, because homologous cycles may cross exactly once. In fact, one can easily construct an arbitrarily large collection of homologous cycles that are pairwise crossing in a graph embedded in the projective plane.



**Figure 2.** Two collections of homologous cycles and their generalized region trees. Left: The cycles are null-homologous. Right: The cycles lie in a non-trivial homology class.

## 5.2 Representing isometric cycles in each homology class

We begin by determining the homology classes of each of the  $O(2^{2g}n)$  isometric cycles in the following manner. Let  $p$  be a simple path, and let  $[p]_h$  denote the bitwise exclusive-or of the homology signatures of its edges. Let  $r$  be the root of any shortest path tree  $T$ . Recall that  $\sigma(r, v)$  denotes the shortest path between  $r$  and  $v$ . It is straightforward to compute  $[\sigma(r, v)]_h$  for every vertex  $v \in V$  in  $O(gn)$  time by iteratively computing signatures in a leafward order. Then, the homology signature of any isometric cycle  $\gamma = \sigma(r, u) \cdot uv \cdot \sigma(v, r)$  can be computed in  $O(g)$  time as  $[\sigma(r, u)]_h \oplus [uv]_h \oplus [\sigma(r, v)]_h$ . We spend  $O(2^{2g}gn) = O(2^{2g}n^2)$  time total computing homology signatures and therefore homology classes. For the remainder of this section, we consider a set of isometric cycles  $\mathcal{C}$  in a single homology class.

Let  $\gamma, \gamma' \in \mathcal{C}$  be two isometric cycles in the same homology class. The combination  $\gamma \oplus \gamma'$  forms the boundary of a subset of faces. That is,  $G \setminus (\gamma \cup \gamma')$  contains at least two components. We represent the cycles in  $\mathcal{C}$  by a tree  $T_{\mathcal{C}}$  where each edge  $e$  of  $T_{\mathcal{C}}$  corresponds to a cycle  $\gamma(e) \in \mathcal{C}$  and each node  $v$  in  $T_{\mathcal{C}}$  corresponds to a subset  $F(v) \in (F \cup \{f_{\infty}\})$ ; specifically, the nodes correspond to sets of faces in the components of  $G \setminus \mathcal{C}$ . This tree generalizes the *region tree* defined by Borradaile, Sankowski and Wulff-Nilsen for planar graphs [4] to more general orientable surface-embedded graphs. We also designate a single representative cycle  $\gamma(\mathcal{C})$  of  $\mathcal{C}$  and pre-compute its cycle signature  $[\gamma(\mathcal{C})]$  for use in our basis cycle finding procedure. See Figure 2.

We describe here the construction of  $T_{\mathcal{C}}$ . Suppose the cycles of  $\mathcal{C}$  have non-trivial homology. Initially,  $T_{\mathcal{C}}$  is a single vertex with one (looping) edge to itself (we will guarantee  $T_{\mathcal{C}}$  is a tree later). Let  $\gamma_0$  be an arbitrary cycle in  $\mathcal{C}$ . We compute  $G' = G \setminus \gamma_0$ . For the one vertex  $v$  of  $T_{\mathcal{C}}$ , we set  $F(v) = F \cup \{f_{\infty}\}$  and for the one edge  $e$ , we set  $\gamma(e) = \gamma_0$ . We will iteratively add additional edges to  $T_{\mathcal{C}}$  corresponding to cycles in  $\mathcal{C}$  before removing the looping edge  $e$ , essentially unfolding  $T_{\mathcal{C}}$  into a tree in the process.

We maintain the invariants that every component of  $G'$  is bounded by two cycles of  $\mathcal{C}$  (initially the cycle  $\gamma_0$  is used twice), each vertex of  $T_{\mathcal{C}}$  is associated with all faces in one component of  $G'$  (possibly including  $f_{\infty}$ ), and each edge  $e$  in  $T_{\mathcal{C}}$  is associated with the cycle in  $\mathcal{C}$  bounding the faces for the two vertices incident to  $e$ . Assuming these invariants are maintained, and because cycles in  $\mathcal{C}$  do not cross, each cycle in  $\mathcal{C}$  lies entirely within some

component of  $G'$ . For each cycle  $\gamma \in \mathcal{C} \setminus \{\gamma_0\}$ , we set  $G' := G' \not\sim \gamma$ , subdivide the vertex associated with the faces of  $C$ 's component, associate the two sets of faces created in  $G'$  with the two new vertices of  $T_{\mathcal{C}}$ , and associate the new edge of  $T_{\mathcal{C}}$  with  $\gamma$ .

Let  $r$  be the vertex of  $T_{\mathcal{C}}$  associated with  $f_{\infty}$ . We set  $\gamma(\mathcal{C})$  to be  $\gamma(e)$ , remove  $e$  from  $T_{\mathcal{C}}$ , and root  $T_{\mathcal{C}}$  at  $r$ . Observe that  $T_{\mathcal{C}}$  has exactly one leaf other than  $r$ .

The procedure is somewhat different if cycles in  $\mathcal{C}$  have trivial homology. Initially,  $T_{\mathcal{C}}$  is a single vertex with no incident edges. For the one vertex  $v$  of  $T_{\mathcal{C}}$ , we set  $F(v) = F \cup \{f_{\infty}\}$ . We set  $G' = G$  initially. We relax our invariants so every component of  $G'$  is bounded by zero or more cycles of  $\mathcal{C}$ , each vertex of  $T_{\mathcal{C}}$  is associated with all faces in one component of  $G'$  (possibly including  $f_{\infty}$ ), and each edge  $e$  in  $T_{\mathcal{C}}$  is associated with the cycle in  $\mathcal{C}$  separating the faces for the two vertices incident to  $e$ . Again, each cycle in  $\mathcal{C}$  lies entirely within some component of  $G'$  at each point in time. For each cycle  $\gamma \in \mathcal{C}$ , we set  $G' := G' \not\sim \gamma$ . We replace the vertex  $v$  associated with the faces of  $C$ 's component with two vertices  $v_1$  and  $v_2$  sharing an edge  $e$ . We associate the new edge  $e$  with  $\gamma$ . We associate the two sets of faces created in  $G'$  with the two new vertices of  $T_{\mathcal{C}}$ . For each edge  $e'$  originally incident to  $v$ , we connect  $e'$  to  $v_1$  if  $\gamma(e')$  is incident to faces of  $v_1$  and connect  $e'$  to  $v_2$  otherwise.

We observe  $T_{\mathcal{C}}$  is a tree; otherwise, there would be a cycle in  $T_{\mathcal{C}}$  passing through some edge  $e$ . The faces associated with vertices of that cycle would contain a walk in  $G$  passing from one side of  $\gamma(e)$  to the other, implying  $\gamma(e)$  is non-separating. We root  $T_{\mathcal{C}}$  at  $r$  and let  $\gamma(\mathcal{C})$  be an arbitrary cycle.

In both cases above, computing  $G' \not\sim \gamma$  for one cycle  $\gamma$  takes  $O(n)$  time. Therefore, we can compute  $T_{\mathcal{C}}$  in  $O(n^2)$  total time.

### 5.3 Selecting an isometric cycle from a homology class

Let  $S$  be an  $(m - n + 1)$ -bit support vector. We describe a procedure to compute  $\langle S, [\gamma] \rangle$  for every isometric cycle  $\gamma$  in  $G$  in  $O(2^{2g}n)$  time. Using this procedure, we can easily return the minimum-weight cycle such that  $\langle S, [\gamma] \rangle = 1$ .

We begin describing the procedure for cycles in the trivial homology class. Let  $\mathcal{C}$  be the collection of null-homologous isometric cycles computed above, and let  $T_{\mathcal{C}}$  be the tree computed for this set. Consider any edge  $e$  of  $T_{\mathcal{C}}$ . The first  $2g$  bits of  $[\gamma(e)]$  are equal to 0, because any co-cycle crosses a cut in the dual an even number of times. Cycle  $\gamma(e)$  bounds a subset of faces  $F'$ . In particular,  $F'$  is the set of faces associated with vertices lying *below*  $e$  in  $T_{\mathcal{C}}$ . The  $i$ th bit of  $[\gamma(e)]$  is 1 if and only if  $p_i$  crosses cut  $F'$  an odd number of times; in other words, the  $i$ th bit is 1 if and only if  $f_i \in F'$ .

We compute  $\langle S, [\gamma] \rangle$  for every cycle  $\gamma \in \mathcal{C}$  in  $O(n)$  time by essentially walking up  $T_{\mathcal{C}}$  in the following manner. For each edge  $e$  in  $T_{\mathcal{C}}$  going to a leaf  $v$ , we maintain a bit  $z$  initially equal to 0 and iterate over each face  $f_i \in F(v)$ . If the  $i$ th bit of  $S$  is equal to 1 then we flip  $z$ . After going through all the faces in  $F(v)$ ,  $z$  is equal to  $\langle S, [\gamma(e)] \rangle$ .

We then iterate up the edges of  $T_{\mathcal{C}}$  toward the root. For each edge  $e$ , we let  $v$  be the lower endpoint of  $e$  and set bit  $z$  equal to the exclusive-or over all  $\langle S, \gamma(e') \rangle$  for edges  $e'$  lying below  $v$ . We then iterate over the faces of  $F(v)$  as before and set  $\langle S, [\gamma(e)] \rangle$  equal to  $z$  as

before. We iterate over every face of  $G$  at most once during this procedure, so it takes  $O(n)$  time total.

Now, consider the set of isometric cycles  $\mathcal{C}$  for some non-trivial homology class. Consider any edge  $e$  of  $T_{\mathcal{C}}$ . Once again, the first  $2g$  bits of  $[\gamma(e)]$  are determined entirely by the homology class, so we only need consider the remaining bits when determining  $\langle S, [\gamma(e)] \rangle$  for different choices of  $e$  in  $T_{\mathcal{C}}$ . Let  $F'$  be the subset of faces bounded by  $\gamma(\mathcal{C}) \oplus \gamma(e)$ . The  $i$ th bit of  $[\gamma(e)]$  disagrees with the  $i$ th bit of  $[\gamma(\mathcal{C})]$  if and only if path  $p_i$  crosses dual cut  $F'$  an odd number of times; in other words, the  $i$ th bits differ if and only if  $f_i \in F'$ . By construction,  $\gamma(\mathcal{C})$  lies on the boundary of  $F(r)$  and  $F(v)$  where  $r$  and  $v$  are the root and other leaf of  $T_{\mathcal{C}}$  respectively. Root  $r$  is the only node of  $T_{\mathcal{C}}$  associated with  $f_{\infty}$ . We conclude the  $i$ th bit of  $[\gamma(e)]$  disagrees with  $[\gamma(\mathcal{C})]$  if and only if  $f_i$  is associated with a vertex lying below  $e$  in  $T_{\mathcal{C}}$ .

We again walk up  $T_{\mathcal{C}}$  to compute  $\langle S, [\gamma] \rangle$  for every cycle  $\gamma \in \mathcal{C}$ . Recall that  $[\gamma(\mathcal{C})]$  is precomputed and stored with  $T_{\mathcal{C}}$ . For each edge  $e$  of  $T_{\mathcal{C}}$  in rootward order, let  $v$  be the lower endpoint of  $e$ . Let  $e'$  be the edge lying below  $e$  in  $T_{\mathcal{C}}$  if it exists (recall that  $T_{\mathcal{C}}$  has exactly one leaf other than its root as cycles in  $\mathcal{C}$  have non-trivial homology). If  $e'$  does not exist, we denote  $\gamma(e')$  as  $\gamma(\mathcal{C})$ . We set  $z$  equal to  $\langle S, \gamma(e') \rangle$ . We then iterate over the faces of  $F(v)$  as before, flipping  $z$  once for every bit  $i$  where  $f_i \in F(v)$  and bit  $i$  of  $S$  is equal to 1. We set  $\langle S, \gamma(e) \rangle := z$ . As before, we consider every face at most once, so walking up  $T_{\mathcal{C}}$  takes  $O(n)$  time.

We have shown the following lemma, which concludes the discussion of our minimum cycle basis algorithm.

**Lemma 5.5.** *Let  $G$  be a graph with  $n$  vertices,  $m$  edges, and  $\ell$  faces cellularly embedded in an orientable surface of genus  $g$  such that  $m = O(n)$  and  $\ell = O(n)$ . We can preprocess  $G$  in  $O(2^{2g}n^2)$  time so that for any  $(m - n + 1)$ -bit support vector  $S$  we can compute the minimum-weight cycle  $\gamma$  such that  $\langle S, \gamma \rangle = 1$  in  $O(2^{2g}n)$  time.*

**Theorem 5.6.** *Let  $G$  be a graph with  $n$  vertices and  $m$  edges, cellularly embedded in an orientable surface of genus  $g$ . We can compute a minimum-weight cycle basis of  $G$  in  $O(n^{\omega} + 2^{2g}n^2 + m)$  time.*

## 5.4 Improving the time for computing and representing isometric cycles

Here we discuss ways in which we can improve the running time for finding and representing isometric cycles using known techniques, thereby isolating the bottleneck of the algorithm to updating the support vectors and computing  $\gamma_j$ .

The set and representation of isometric cycles can be computed recursively using  $O(\sqrt{gn})$  balanced separators (e.g. [1]) as inspired by Wulff-Nilsen [36]. Briefly, given a set  $S$  of  $O(\sqrt{gn})$  separator vertices (for a graph of bounded genus), find all the isometric cycles in each component of  $G \setminus S$  and represent these isometric cycles in at most  $2^{2g}$  region trees per component, as described above. Merging the region trees for different components of  $G \setminus S$  is relatively simple since different sets of faces are involved. It remains to compute the set

of isometric cycles that contain vertices of  $S$  and add them to their respective region trees. First note that a cycle that is isometric in  $G$  and does not contain a vertex of  $S$  is isometric in  $G \setminus S$ , but a cycle that is isometric in  $G \setminus S$  may not be isometric in  $G$ , so indeed we are computing a superset of the set of isometric cycles via this recursive procedure. However, it is relatively easy to show that an isometric cycle of  $G \setminus S$  can cross an isometric cycle of  $G$  at most once, so, within a given homology class, isometric cycles will nest and be representable by a region tree.

To compute the set of isometric cycles that intersect vertices of  $S$ , we first compute shortest path trees rooted at each of the vertices of  $S$ , generating the Horton cycles rooted at these vertices; this procedure takes  $O(\sqrt{gn} \cdot n)$  time using the linear time shortest path algorithm for graphs excluding minors of sub-linear size [34]. We point out that the algorithm of Amaldi *et al.* [2] works by identifying Horton cycles that are not isometric and by identifying, among different Horton-cycle representations of a given isometric cycle, one representative; this can be done for a subset of Horton cycles, such as those rooted in vertices of  $S$ , and takes time proportional to the size of the representation of the Horton cycles (i.e., the  $O(\sqrt{n})$  shortest path trees, or  $O(\sqrt{gn}^{1.5})$ ).

For a given homology class of cycles, using the shortest-path tree representation of the isometric cycles, we can identify those isometric cycles in that homology class by computing the homology signature of root-to-node paths in the shortest path tree as before; this process can be done in  $O(\sqrt{gn}^{1.5})$  time. We must now add these cycles to the corresponding region tree. Borradaile, Sankowski and Wulff-Nilsen [4] describe a method for adding  $n$  cycles to a region tree in  $O(n \text{ poly log } n)$  time that is used in their minimum cycle basis algorithm for planar graphs; this method will generalize to surfaces for nesting cycles. Therefore computing the homology classes of these isometric cycles and adding these isometric cycles to the region trees takes a total of  $O(2^{2g} \sqrt{gn}^{1.5})$  time.

In total, this recursive method for computing and building a representation of a superset of the isometric cycles takes time given by the recurrence relation  $T(n) = 2T(n/2) + O(2^{2g} \sqrt{gn}^{1.5})$  or  $O(2^{2g} \sqrt{gn}^{1.5})$  time.

## 6 Homology Basis

We now describe our algorithm for computing a minimum homology basis. Our algorithm works for both orientable and non-orientable surfaces, although we assume without loss of generality that the surface contains at least one boundary component. At a high level, our algorithm for minimum homology bases is very similar to our algorithm for minimum cycle bases. As before, our algorithm incrementally adds simple cycles  $\gamma_1, \dots, \gamma_\beta$  to the minimum homology basis by maintaining a set of  $\beta$  support vectors  $S_1, \dots, S_\beta$  such that the following hold:

- The support vectors form a basis for  $\mathbb{Z}_2^\beta$ .
- When the algorithm is about to compute the  $j$ th cycle  $\gamma_j$  for the minimum homology basis,  $\langle S_j, [\gamma_{j'}]_h \rangle = 0$  for all  $j' < j$ .



Our algorithm chooses for  $\gamma_j$  the minimum-weight simple cycle  $\gamma$  such that  $\langle S_j, [\gamma]_h \rangle = 1$ . The following lemma has essentially the same proof as Lemma 4.1.

**Lemma 6.1.** *Let  $S$  be a  $\beta$ -bit vector with at least one bit set to 1, and let  $\eta$  be the minimum-weight cycle such that  $\langle S, [\eta]_h \rangle = 1$ . Then,  $\eta$  is a member of the minimum homology basis.*

As before, our algorithm updates the support vectors and computes minimum homology basis cycles in a recursive manner. We define  $\text{extend}(j, k)$  and  $\text{update}(j, k)$  as before, using homology signatures in place of cycle signatures when applicable. Our algorithm runs  $\text{extend}(1, \beta)$  to compute the minimum homology basis.

The one crucial difference between our minimum cycle basis and minimum homology basis algorithms is the procedure we use to find each minimum homology basis cycle  $\gamma_j$  given support vector  $S_j$ . The homology basis procedure takes  $O(\beta^2 n \log n)$  time instead of  $O(2^{2g} n)$  time, and it requires no preprocessing step. We describe the procedure in Sections 6.1 and 6.2.

The procedure  $\text{update}(j, k)$  takes only  $O(\beta k^{\omega-1})$  time in our minimum homology basis algorithm, because signatures have length  $\beta$ . Therefore, we can bound the running time of  $\text{extend}(j, k)$  using the following recurrence:

$$T(k) = \begin{cases} 2T(k/2) + O(\beta k^{\omega-1}) & \text{if } k > 1 \\ O(\beta^2 n \log n) & \text{if } k = 1 \end{cases}$$

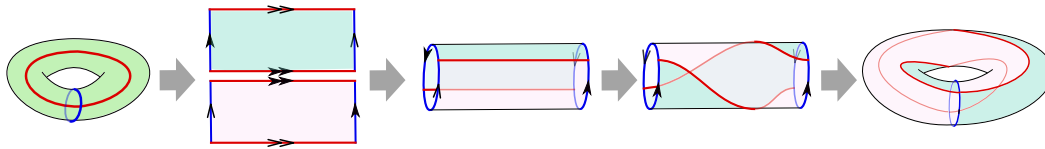
The total time spent in calls to  $\text{extend}(j, k)$  where  $k > 1$  is  $O(\beta k^{\omega-1})$ . The total time spent in calls to  $\text{extend}(j, 1)$  is  $O(\beta^2 k n \log n)$ . Therefore,  $T(k) = O(\beta k^{\omega-1} + \beta^2 k n \log n)$ . The running time of our minimum homology basis algorithm<sup>3</sup> (after sparsifying  $G$ ) is  $T(\beta) = O(\beta^3 n \log n) = O((g + b)^3 n \log n)$ .

## 6.1 Cyclic double cover

In order to compute minimum homology basis cycle  $\gamma_j$ , we lift the graph into a *covering space* known as the **cyclic double cover**. Our presentation of the cyclic double cover is similar to that of Erickson [16]. Erickson describes the cyclic double cover relative to a single simple non-separating cycle in an orientable surface; however, we describe it relative to an arbitrary *set* of non-separating *co-paths* determined by a support vector  $S$ , similar to the homology cover construction of Erickson and Nayyeri [17]. Our construction works for non-orientable surfaces without any special considerations.

Let  $S$  be a  $\beta$ -bit support vector for the minimum homology basis problem as defined above. We define the cyclic double cover relative to  $S$  using a standard *voltage construction* [22, Chapter 4]. Let  $G_S^2$  be the graph whose vertices are pairs  $(v, z)$ , where  $v$  is a vertex of  $G$  and  $z$  is a bit. The edges of  $G_S^2$  are ordered pairs  $(uv, z) := (u, z)(v, z \oplus \langle S, [uv]_h \rangle)$  for

<sup>3</sup>Our minimum homology basis algorithm can be simplified somewhat by having  $\text{extend}(j, k)$  recurse on  $\text{extend}(j, 1)$  and  $\text{extend}(j + 1, k - 1)$  and by using a simpler algorithm for  $\text{update}(j, k)$ . This change will increase the time spent in calls to  $\text{extend}(j, k)$  where  $k > 1$ , but the time taken by calls with  $k = 1$  will still be a bottleneck on the overall run time.



**Figure 3.** Constructing the cyclic double cover. Left to right: A pair of co-cycles  $\Psi$  on the torus  $\Sigma$ ; the surfaces  $(\Sigma', 0)$  and  $(\Sigma', 1)$ ; identifying copies of one co-cycle; preparing to identify copies of the other co-cycle; the cyclic double cover.

all edges  $uv$  of  $G$  and bits  $z$ . Let  $\pi : G_S^2 \rightarrow G$  denote the **covering map**  $\pi(v, z) = v$ . The **projection** of any vertex, edge, or path in  $G_S^2$  is the natural map to  $G$  induced by  $\pi$ . We say a vertex, edge, or path  $p$  in  $G$  **lifts** to  $p'$  if  $p$  is the projection of  $p'$ . A closed path in  $G_S^2$  is defined to bound a face (be a boundary component) of  $G_S^2$  if and only if its projection with regard to  $\pi$  bounds a face (is a boundary component) of  $G$ . This construction defines an embedding of  $G_S^2$  onto a surface  $\Sigma_S^2$ ; we will prove  $G_S^2$  and  $\Sigma_S^2$  are connected shortly.

We can also define  $G_S^2$  in a more topologically intuitive way as follows. Let  $\Psi$  be a set of co-paths which contains each co-path  $p_i$  for which the  $i$ th bit of  $S$  is equal to 1. Let  $\Sigma'$  be the surface obtained by cutting  $\Sigma$  along the image of each co-path in  $\Psi$ . Note that  $\Sigma'$  may be disconnected. Each co-path  $p_i \in \Psi$  appears as two copies on the boundary of  $\Sigma'$  denoted  $p_i^-$  and  $p_i^+$  (note that  $p_i^-$  and  $p_i^+$  may themselves be broken into multiple components). Create two copies of  $\Sigma'$  denoted  $(\Sigma', 0)$  and  $(\Sigma', 1)$ , and let  $(p_i^-, z)$  and  $(p_i^+, z)$  denote the copies of  $p_i^-$  and  $p_i^+$  in surface  $(\Sigma', z)$ . For each co-path  $p_i \in \Psi$ , we identify  $(p_i^-, 0)$  with  $(p_i^-, 1)$  and we identify  $(p_i^+, 1)$  with  $(p_i^+, 0)$ , creating the surface  $\Sigma_S^2$  and the graph  $G_S^2$  embedded on  $\Sigma_S^2$ . See Figure 3.

The first three of the following lemmas are immediate.

**Lemma 6.2.** *Let  $\gamma$  be any simple cycle in  $G$ , and let  $s$  be any vertex of  $\gamma$ . Then  $\gamma$  is the projection of a unique path in  $G_S^2$  from  $(s, 0)$  to  $(s, \langle S, [\gamma]_h \rangle)$ .*

**Lemma 6.3.** *Every lift of a shortest path in  $G$  is a shortest path in  $G_S^2$ .*

**Lemma 6.4.** *Let  $\gamma$  be the minimum-weight simple cycle of  $G$  such that  $\langle S, [\gamma]_h \rangle = 1$ , and let  $s$  be any vertex of  $\gamma$ . Then  $\gamma$  is the projection of the shortest path in  $G_S^2$  from  $(s, 0)$  to  $(s, 1)$ .*

**Lemma 6.5.** *The cyclic double cover  $G_S^2$  is connected.*

**Proof:** There exists some simple cycle  $\gamma$  in  $G$  such that  $\langle S, [\gamma]_h \rangle = 1$ . Let  $s$  be any vertex of  $\gamma$ . Let  $v$  be any vertex of  $G$ . We show there exists a path from  $(v, z)$  to  $(s, 0)$  in  $G_S^2$  for both bits  $z$ . There exists a path from  $v$  to  $s$  in  $G$  so there is a path from  $(v, z)$  to one of  $(s, 0)$  or  $(s, 1)$  in  $G_S^2$ . The other of  $(s, 0)$  or  $(s, 1)$  may be reached by following the lift of  $\gamma$ .  $\square$

Observe that  $G_S^2$  has  $2n$  vertices and  $2m$  edges. Each co-path  $p_i$  shares an even number of edges with each face of  $G$ . By Lemma 6.2, both lifts of any face  $f$  to  $G_S^2$  are

cycles; in particular both lifts are faces. However, there may be one or more boundary cycles  $\gamma$  of  $G$  such that  $\langle S, [\gamma]_h \rangle = 1$ . It takes both lifts of such a cycle  $\gamma$  to make a single boundary component in  $G_S^2$ . We conclude  $G_S^2$  contains  $2\ell$  faces and between  $b$  and  $2b$  boundary cycles. Surface  $\Sigma_S^2$  has Euler characteristic  $2n - 2m + 2\ell = 2\chi$ . It is non-orientable if and only if there exists a one-sided cycle  $\eta$  such that  $\langle S, [\eta]_h \rangle = 0$ . If both  $\Sigma$  and  $\Sigma_S^2$  are non-orientable, then  $\Sigma_S^2$  has genus at most  $2g + b$ . If only  $\Sigma$  is non-orientable, then  $\Sigma_S^2$  has genus at most  $g + b/2 - 1$ . If both surfaces are orientable, then  $\Sigma_S^2$  has genus at most  $2g + b/2 - 1$ . In all three cases, the genus is at most  $O(\beta)$ .

## 6.2 Selecting homology basis cycles

Let  $S$  be any  $\beta$ -bit support vector. We now describe our algorithm to select the minimum-weight cycle  $\gamma$  such that  $\langle S, [\gamma]_h \rangle = 1$ . Our algorithm is based on one by Erickson and Nayyeri [17] for computing minimum-weight cycles in arbitrary homology classes, except we use the cyclic double cover instead of their  $\mathbb{Z}_2$ -homology cover. We have the following lemma. While it was shown with orientable surfaces in mind, the proof translates verbatim to the non-orientable case.

**Lemma 6.6 (Erickson and Nayyeri [17, Lemma 5.1]).** *In  $O(n \log n + \beta n)$  time, we can construct<sup>4</sup> a set  $\Pi$  of  $O(\beta)$  shortest paths in  $G$ , such that every non-null-homologous cycle in  $G$  intersects at least one path in  $\Pi$ .*

Let  $G_S^2$  be the cyclic double cover of  $G$  with regard to  $S$ . Our algorithm constructs  $G_S^2$  in  $O(\beta n)$  time.

Suppose our desired cycle  $\gamma$  intersects shortest path  $\sigma \in \Pi$  at some vertex  $s$ . By Lemma 6.4, simple cycle  $\gamma$  is the projection of the shortest path in  $G_S^2$  from  $(s, 0)$  to  $(s, 1)$ . Let  $\hat{\gamma}$  be this shortest path in  $G_S^2$ . Let  $\hat{\sigma}$  be the lift of  $\sigma$  to  $G_S^2$  that contains vertex  $(s, 0)$ . By Lemma 6.3, path  $\hat{\sigma}$  is also a shortest path in  $G_S^2$ . If  $\hat{\gamma}$  uses any other vertex  $(v, z)$  of  $\hat{\sigma}$  other than  $(s, 0)$ , then it can use the entire subpath of  $\hat{\sigma}$  between  $(s, 0)$  and  $(v, z)$ .

Now, consider the surface  $\Sigma_S^2 \setminus \hat{\sigma}$  which contains a single face bounded by two copies of  $\hat{\sigma}$  we denote  $\hat{\sigma}^-$  and  $\hat{\sigma}^+$ . For each vertex  $(v, z)$  on  $\hat{\sigma}$ , let  $(v, z)^-$  and  $(v, z)^+$  denote its two copies on  $\hat{\sigma}^-$  and  $\hat{\sigma}^+$  respectively. From the above discussion, we see  $\hat{\gamma}$  is a shortest path in  $\Sigma_S^2 \setminus \hat{\sigma}$  from one of  $(s, 0)^-$  or  $(s, 0)^+$  to  $(s, 1)$ .

To find  $\gamma$ , we use the following generalization of Klein's [29] multiple-source shortest path algorithm:

**Lemma 6.7 (Cabello *et al.* [8]).** *Let  $G$  be a graph with  $n$  vertices, cellularly embedded in a surface of genus  $g$ , and let  $f$  be any face of  $G$ . We can preprocess  $G$  in  $O(gn \log n)$  time and  $O(n)$  space so that the length of the shortest path from any vertex incident to  $f$  to any other vertex can be retrieved in  $O(\log n)$  time.*

Our algorithm iterates over the  $O(\beta)$  shortest paths present in  $\Pi$ . For each such path  $\sigma$ , it computes a lift  $\hat{\sigma}$  in  $G_S^2$ , cuts  $\Sigma_G^2$  along  $\hat{\sigma}$ , and runs the multiple-source shortest

<sup>4</sup>We only need to construct  $\Pi$  once for the entire minimum homology basis algorithm, but constructing it once per basis cycle does not affect the overall run time.

path procedure of Lemma 6.7 to find the shortest path from some vertex  $(s, z)^\pm$  on  $\hat{\sigma}^\pm$  to  $(s, z \oplus 1)$ . Each shortest path it finds projects to a closed path  $\gamma'$  such that  $\langle S, [\gamma']_h \rangle = 1$ . By the above discussion, the shortest such projection can be chosen for  $\gamma$ . Running the multiple-source shortest path procedure  $O(\beta)$  times on a graph of genus  $O(\beta)$  takes  $O(\beta^2 n \log n)$  time total. We conclude the discussion of our minimum homology basis algorithm.

**Lemma 6.8.** *Let  $G$  be a graph with  $n$  vertices,  $m$  edges, and  $\ell$  faces cellularly embedded in a surface of genus  $g$  such that  $m = O(n)$  and  $\ell = O(n)$ . For any  $\beta$ -bit support vector  $S$  we can compute the minimum-weight cycle  $\gamma$  such that  $\langle S, \gamma \rangle = 1$  in  $O(\beta^2 n \log n)$  time.*

**Theorem 6.9.** *Let  $G$  be a graph with  $n$  vertices and  $m$  edges, cellularly embedded in an orientable or non-orientable surface of genus  $g$  with  $b$  boundary components. We can compute a minimum-weight homology basis of  $G$  in  $O((g + b)^3 n \log n + m)$  time.*

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