Spanning Paths in Fibonacci-sum Graphs

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Abstract

Motivated by a problem posed by Barwell, we apply graph theory to determine all n for which the numbers $1, \ldots, n$ can be ordered so that the sum of any two consecutive terms is a Fibonacci number. We prove that such an ordering exists if and only if n is 9, 11, a Fibonacci number, or one less than a Fibonacci number. For each such n, we also prove that at most two such orderings exist, up to symmetry.

In this paper, we consider a problem posed by Barwell [1]. Barwell asked for an ordering of the numbers $1, \ldots, 34$ such that any two consecutive terms sum to a Fibonacci number. We attack this problem using graph theory, by defining a graph with vertices $\{1, \ldots, 34\}$ and edges $\{uv : u + v \text{ is a Fibonacci number}\}$. An ordering of the desired form then corresponds to a spanning path in the graph, i.e. a path that visits each vertex. Using this approach, we solve a more general problem by determining all n such that the numbers $1, \ldots, n$, have such an ordering: this holds if and only if n is 9, 11, a Fibonacci number, or one less than a Fibonacci number. We also prove that for each such n there are at most two such orderings (up to symmetry).

We write $[n]$ to denote $\{1, 2, \ldots, n\}$. When discussing the Fibonacci numbers, we adopt the usual convention that $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. We will, at several points, use the well-known observations that F_k is even if and only if $3|k$ and that any two consecutive Fibonacci numbers are relatively prime.

We begin by formally defining the graph we will use to model Barwell's original problem.

Definition 1. *For $n \geq 1$, the* Fibonacci-sum graph on $[n]$, *denoted G_n, is the graph with vertex set $[n]$ and edge set $\{uv : u + v = F_i$ for some $i\}$. We freely treat the elements of $[n]$ either as vertices of G_n or as integers. The* sum *of an edge in G_n is the sum of its endpoints.*

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As suggested above, a spanning path in G_n corresponds to an ordering of $[n]$ such that any two consecutive terms sum to a Fibonacci number. When $n \leq 4$ the graph G_n is itself a path, so usually we restrict our attention to the case $n \geq 5$. To attack Barwell's problem, we will show that when $k \geq 5$, the graph G_{F_k} has a spanning path.

Theorem 1. *If $k \geq 5$, then the graph G_{F_k} has a spanning path.*

Proof. Let P_k be the spanning subgraph of G_{F_k} containing precisely those edges having sums in $\{F_{k-1}, F_k, F_{k+1}\}$. We claim that in fact P_k is a path. To prove this, it suffices to show that P_k contains no cycles and that two vertices of G_{F_k} have degree 1 in P_k, while the rest have degree 2.

To see that each vertex of G_{F_k} has degree at most 2 in P_k, it suffices to show that each vertex lies on at most two edges having sums in $\{F_{k-1}, F_k, F_{k+1}\}$. This follows from the observation that no vertex in $\{1, \ldots, F_{k-1} - 1\}$ lies on an edge having sum F_{k+1}, and no vertex in $\{F_{k-1}, \ldots, F_k\}$ lies on an edge having sum F_{k-1}. Conversely, each vertex smaller than F_{k-1} does lie on an edge having sum F_{k-1}, except perhaps for the vertex $F_{k-1}/2$ when F_{k-1} is even. Likewise, each vertex smaller than F_k lies on an edge having sum F_k, except for the vertex $F_k/2$ when F_k is even. Finally, each vertex in $\{F_{k-1}, \ldots, F_k\}$ lies on an edge having sum F_{k+1}, except for the vertex $F_{k+1}/2$ when F_{k+1} is even. Thus each vertex of G_{F_k} has degree at least 2 in P_k, except possibly for $F_k, F_{k-1}/2, F_k/2$, and $F_{k+1}/2$. The vertex F_k must have degree 1, and the vertices $F_{k-1}/2$, $F_k/2$, and $F_{k+1}/2$ have degree 1 precisely when F_{k-1}, F_k, and F_{k+1}, respectively, are even. Since exactly one of F_{k-1}, F_k, and F_{k+1} is even, P_k has exactly two vertices of degree 1.

We next claim that P_k contains no cycles. Suppose otherwise, and let C be a cycle in P_k. Each vertex of C lies on exactly two edges in P_k; by the arguments in the preceding paragraph, exactly one of these edges must have sum F_k. Thus the edges in C alternate between those having sum F_k and those having sum in $\{F_{k-1}, F_{k+1}\}$. Let α and β denote the numbers of edges in C having sums F_{k-1} and F_{k+1}, respectively; note that C has $\alpha + \beta$ edges with sum F_k. Now let us sum the vertices in C. Since each vertex lies on exactly one edge having sum F_k, the sum is $(\alpha + \beta)F_k$; since each vertex lies on exactly one edge having sum in $\{F_{k-1}, F_{k+1}\}$, the sum is $\alpha F_{k-1} + \beta F_{k+1}$. Thus $(\alpha + \beta)F_k = \alpha F_{k-1} + \beta F_{k+1}$, so $\alpha(F_k - F_{k-1}) = \beta(F_{k+1} - F_k)$; applying the Fibonacci recurrence once on each side yields $\alpha F_{k-2} = \beta F_{k-1}$. Since F_{k-2} and F_{k-1} are relatively prime, it follows that $F_{k-1} | \alpha$ and $F_{k-2} | \beta$; consequently, $\alpha \geq F_{k-1}$ and $\beta \geq F_{k-2}$. Thus $\alpha + \beta \geq F_k$, which is impossible since P_k has only $F_k - 1$ edges. □

We will next show that always the graph G_{F_k} has at most two spanning paths: one is P_k, and the other (when it exists) differs from P_k in only two edges. Thus $[F_k]$ has at most two orderings of the desired form (up to symmetry).

Lemma 2. *Let k be an integer greater than 4, and choose m from $\{k-1, k, k+1\}$ so that F_m is even. In the graph G_{F_k}, the path P_k is the only spanning path whose endpoints are $F_m/2$ and F_k.*

Proof. As argued in the proof of Theorem 1, no vertex in G_{F_k} lies on edges having sums both F_{k-1} and F_{k+1}. From this fact and the observation that $F_m/2$ lies on no edge having sum F_m, it follows that $F_m/2$ has degree 1 in P_k. Likewise, F_k also has degree 1 in P_k, since it is too large to lie on edges having sums in $\{F_{k-1}, F_k\}$. Thus $F_m/2$ and F_k are the endpoints of P_k, as claimed.

Now suppose that G_{F_k} has some other spanning path P with these same endpoints. By definition of P_k, every edge that belongs to P_k but not to P has sum at least F_{k-1}. Similarly, every edge that belongs to P but not to P_k has sum less than F_{k-1}; otherwise, it would belong to P_k. Since at least one edge belongs to P but not to P_k, we have

$$\sum_{uv \text{ in } P_k} (u+v) > \sum_{uv \text{ in } P} (u+v)\,.$$

However, both sums above include $F_m/2$ and F_k once each and every other vertex in G_{F_k} twice. Thus the two sums must be equal, so we have a contradiction. □

Theorem 3. *Let k be an integer greater than 4. If $k \not\equiv 1 \mod 3$, then G_{F_k} has a unique spanning path. If $k \equiv 1 \mod 3$, then G_{F_k} has exactly two spanning paths: one has endpoints F_k and $F_{k-1}/2$, while the other has endpoints F_k and $F_k - F_{k-4}/2$.*

Proof. Since F_k has degree 1 in G_{F_k}, every spanning path has F_k as one endpoint.

If $k \equiv 2 \mod 3$, then $F_{k+1}/2$ has degree 1, since it cannot lie on an edge having sum F_{k+1} and is too large to lie on an edge having sum F_{k-1}. Thus every spanning path has endpoints F_k and $F_{k+1}/2$; by Lemma 2, P_k is the only such path. Similarly, if $k \equiv 0 \mod 3$, then $F_k/2$ has degree 1, and again P_k is the only spanning path.

Finally, suppose $k \equiv 1 \mod 3$. Let $S = \{F_{k-4}/2, F_{k-1}/2, F_{k-1} + F_{k-4}/2, F_k - F_{k-4}/2\}$. Note that S induces a cycle in G_{F_k}. Moreover, of the vertices in S, only $F_{k-4}/2$ has neighbors outside S, so the vertices of S must occur at one end of any spanning path. Every spanning path must enter S at $F_{k-4}/2$ and hence must end either at $F_{k-1}/2$ or at $F_k - F_{k-4}/2$. We can transform any spanning path ending at $F_{k-1}/2$ into one ending at $F_k - F_{k-4}/2$ (and vice-versa) by permuting the last three vertices; Lemma 2 shows that G_{F_k} has only one spanning path of the former type, so it also has only one of the latter type. □

Note that when G_{F_k} has only one spanning path, that path is P_k; when G_{F_k} has two spanning paths, one is P_k, and the other can be obtained from P_k by permuting the last three vertices. In this sense, P_k is "essentially" the only spanning path in G_{F_k}.

For our last result, we determine all n such that G_n has a spanning path; here we do not require that n be a Fibonacci number, although we do make use of our previous results.

Theorem 4. *When $n \geq 5$, the graph G_n has a spanning path if and only if $n = 9$ or $n = 11$ or $n \in \{F_i, F_i - 1\}$ for some i. This spanning path is unique unless $n \in \{F_i, F_i - 1\}$ with $i \equiv 1 \mod 3$, in which case G_n has two spanning paths.*

Proof. Recall that F_i has degree 1 in G_{F_i}. Thus, given a spanning path in G_{F_i}, removing F_i leaves a spanning path in G_{F_i-1}. Likewise, F_{i-1} has degree 1 in G_{F_i-1}; given a spanning path in G_{F_i-1}, adding the edge $F_{i-1}F_i$ yields a spanning path in G_{F_i}. Thus G_{F_i} and G_{F_i-1} have the same number of spanning paths; our claims for G_{F_i-1} now follow by Theorem 3. By inspection, G_9 and G_{11} have unique spanning paths.

Now suppose that n is not 9 or 11 or of the form F_i or $F_i - 1$. Choose k such that $F_{k-1} + 1 \leq n \leq F_k - 2$. Since $n < F_k - 1$, the vertices F_{k-1} and $F_{k-1} + 1$ can lie only on edges having sum F_k, so each has degree 1 in G_n. If $F_{k-1} + 1 < n < F_k - 2$, then $F_{k-1} + 2$ is present and likewise has degree 1, which precludes the existence of a spanning path. Hence we may suppose that $n \in \{F_{k-1} + 1, F_k - 2\}$.

Suppose first that $n = F_{k-1} + 1$, and let P be a spanning path in G_n. The vertex n has degree 1 in G_n, so it must be an endpoint of P; removing it yields a spanning path P' in $G_{F_{k-1}}$. Note that n is adjacent in G_n to $F_{k-2} - 1$, so P' has $F_{k-2} - 1$ as an endpoint. We established in the proof of Theorem 3 that the endpoints of every spanning path in $G_{F_{k-1}}$ lie in $\{F_{k-1}, F_{k-2}/2, F_{k-1}/2, F_k/2, F_{k-1} - F_{k-5}/2\}$. Thus $F_{k-2} - 1$ lies in this set; checking each possibility yields $n \in \{4, 9\}$, contradicting the choice of n.

Suppose instead that $n = F_k - 2$, and let P be a spanning path in G_n. Both F_{k-1} and $F_{k-1}+1$ have degree 1 in G_n, so they must be the endpoints of any spanning path. Moreover, in G_{F_k}, we have F_{k-1} adjacent to F_k and $F_{k-1} + 1$ adjacent to $F_k - 1$. Thus we may extend P to a spanning path P' in G_{F_k} having endpoints F_k and $F_k - 1$. As in the prior case, this implies that $F_k - 1$ lies in $\{F_k, F_{k-1}/2, F_k/2, F_{k+1}/2, F_k - F_{k-4}/2\}$. Checking each possibility yields $n \in \{0, 3, 11\}$, again contradicting the choice of n. □

What if we consider the analogous graphs corresponding to the numbers A_i, where $A_1 = a$, $A_2 = b$, and $A_k = A_{k-1} + A_{k-2}$ for $k \geq 3$? We call such graphs the *generalized Fibonacci-sum graphs*. In the proofs of Theorem 1 and Lemma 2, we only used the Fibonacci recurrence, the fact that every third Fibonacci number is even, and the fact that consecutive Fibonacci numbers are relatively prime; these two facts follow easily for the numbers A_i if we require that a and b be relatively prime. In fact, this requirement is also needed for the existence of a spanning path: if a and b have a nontrivial common divisor d, then the endpoints of every edge in the graph belong to the same congruence class modulo d, so no path contains both 1 and 2. Thus as long as a and b are relatively prime, Theorem 1 and Lemma 2 also hold for the corresponding generalized Fibonacci-sum graphs; Theorems 3 and 4 hold with a few minor, straightforward alterations.

References

[1] B. Barwell, Problem 2732, Problems and conjectures, *Journal of Recreational Mathematics* 34 (2006), no. 3, 220-223.