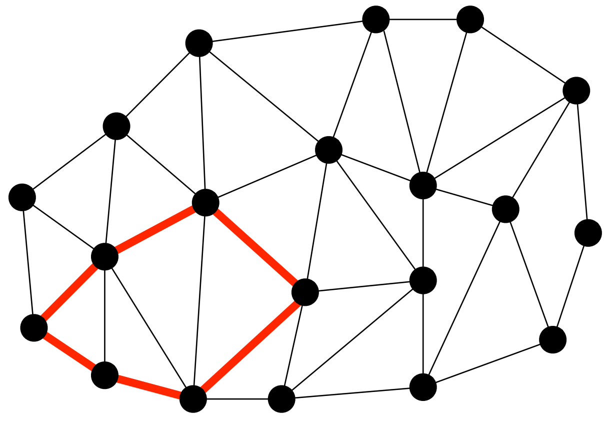
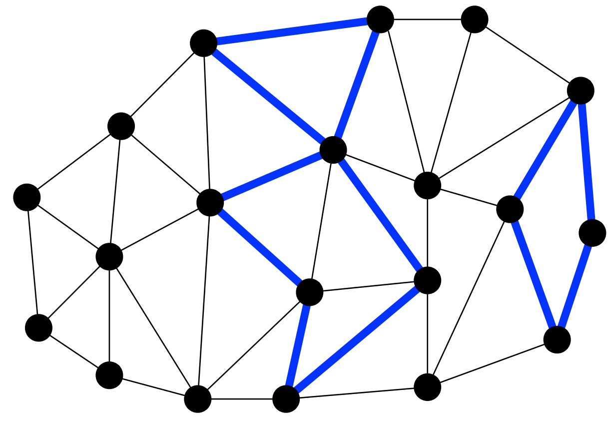




Minimum cycle and homology bases of surface embedded graphs

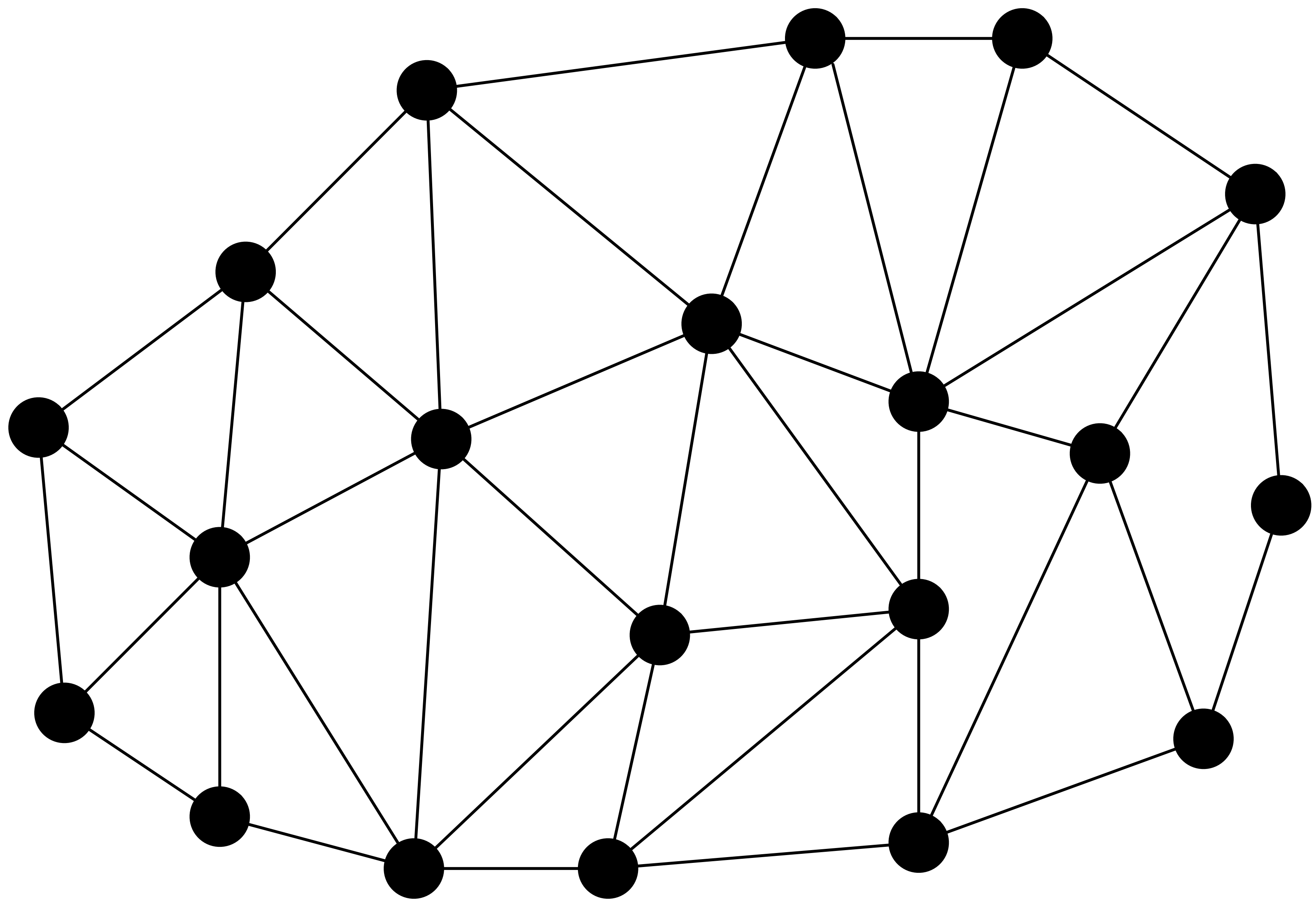
Glencora Borradaile
Oregon State University

Erin Wolf Chambers
St. Louis University

Kyle Fox
Duke University

Amir Nayyeri
Oregon State University

Cycles



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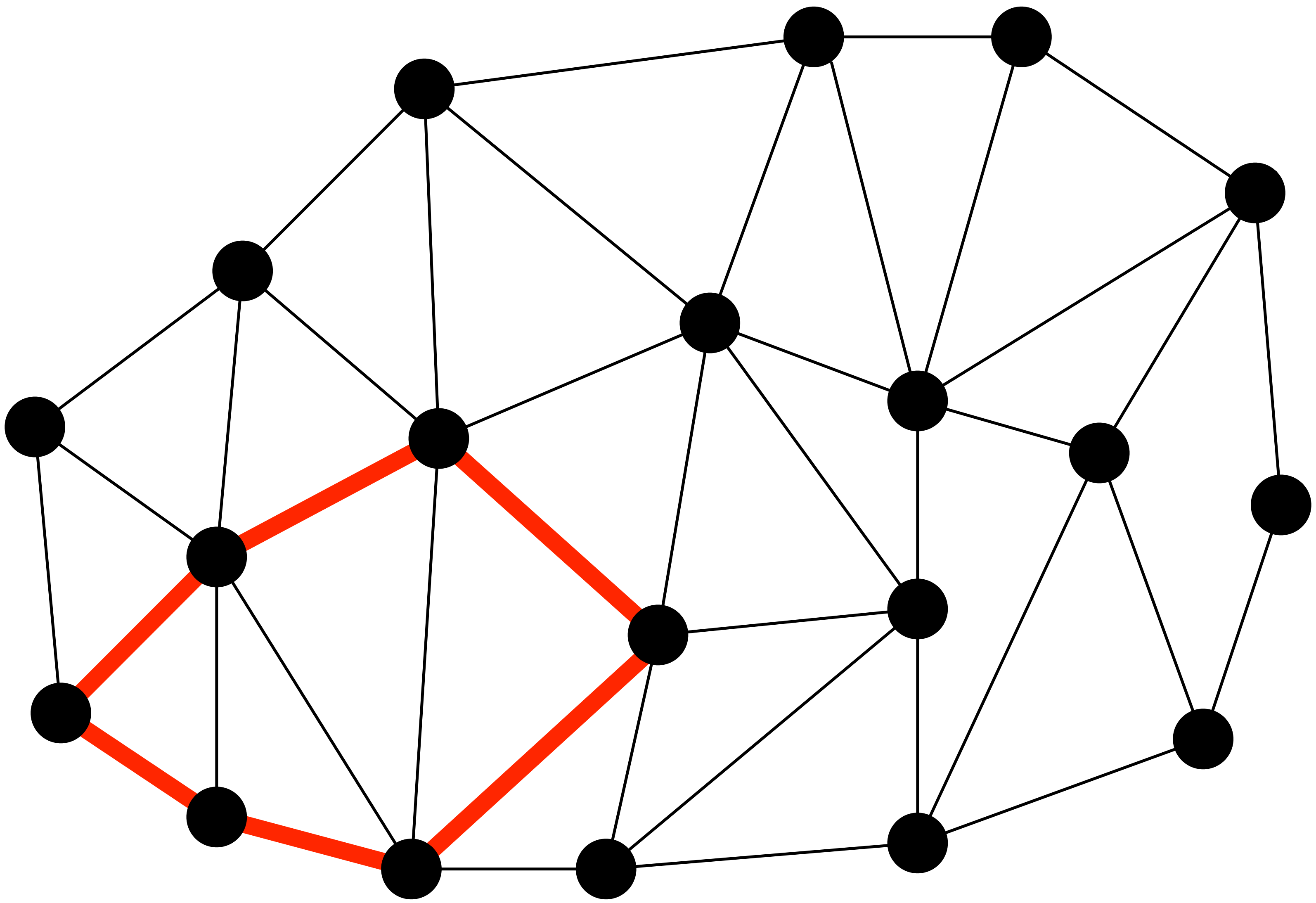

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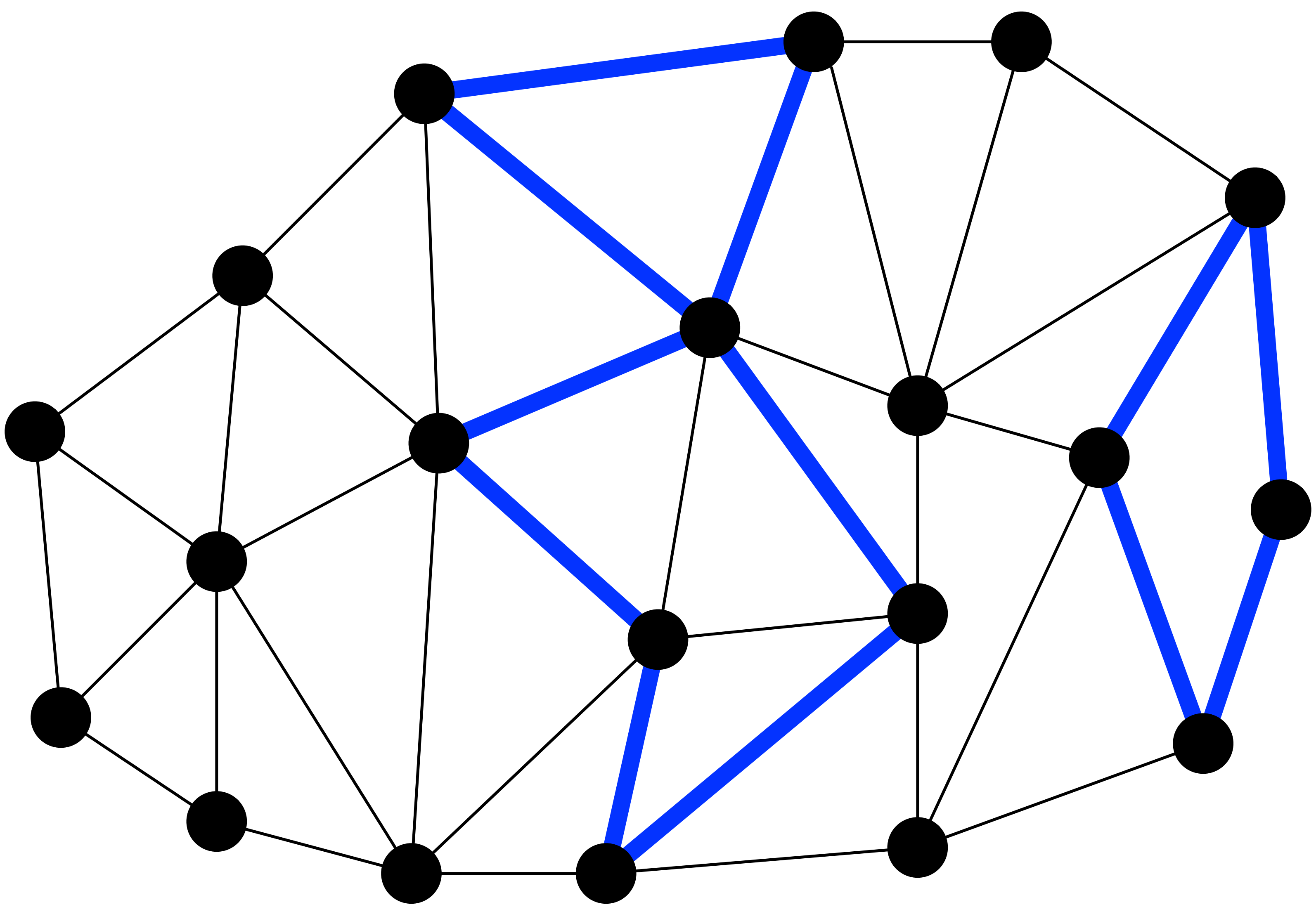

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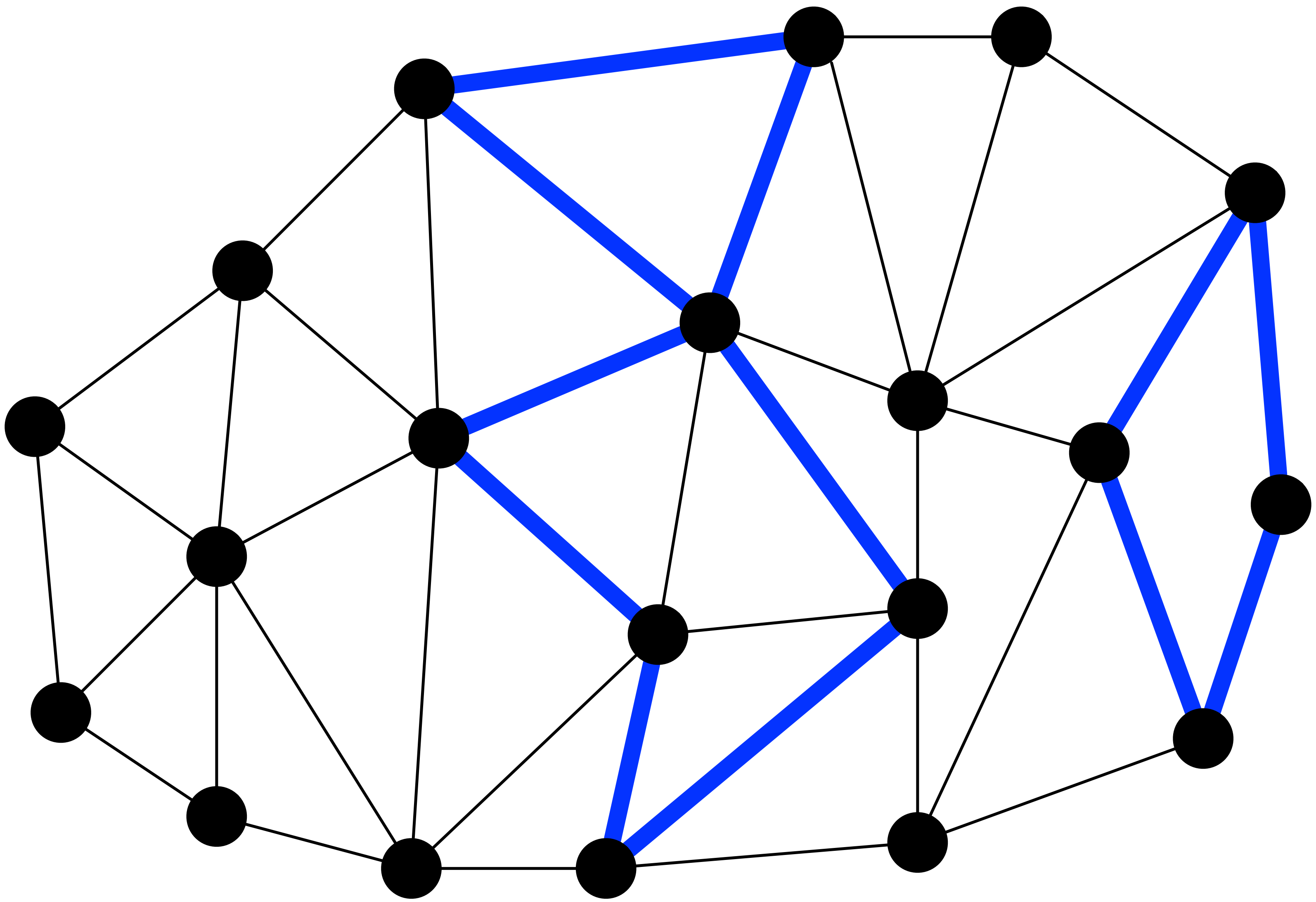
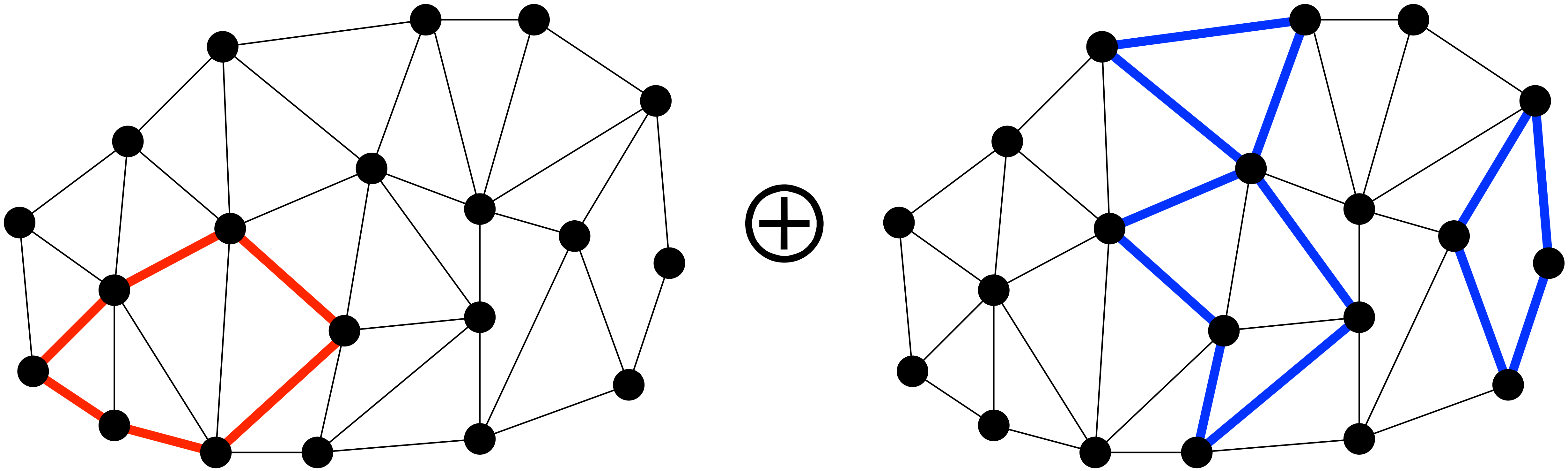
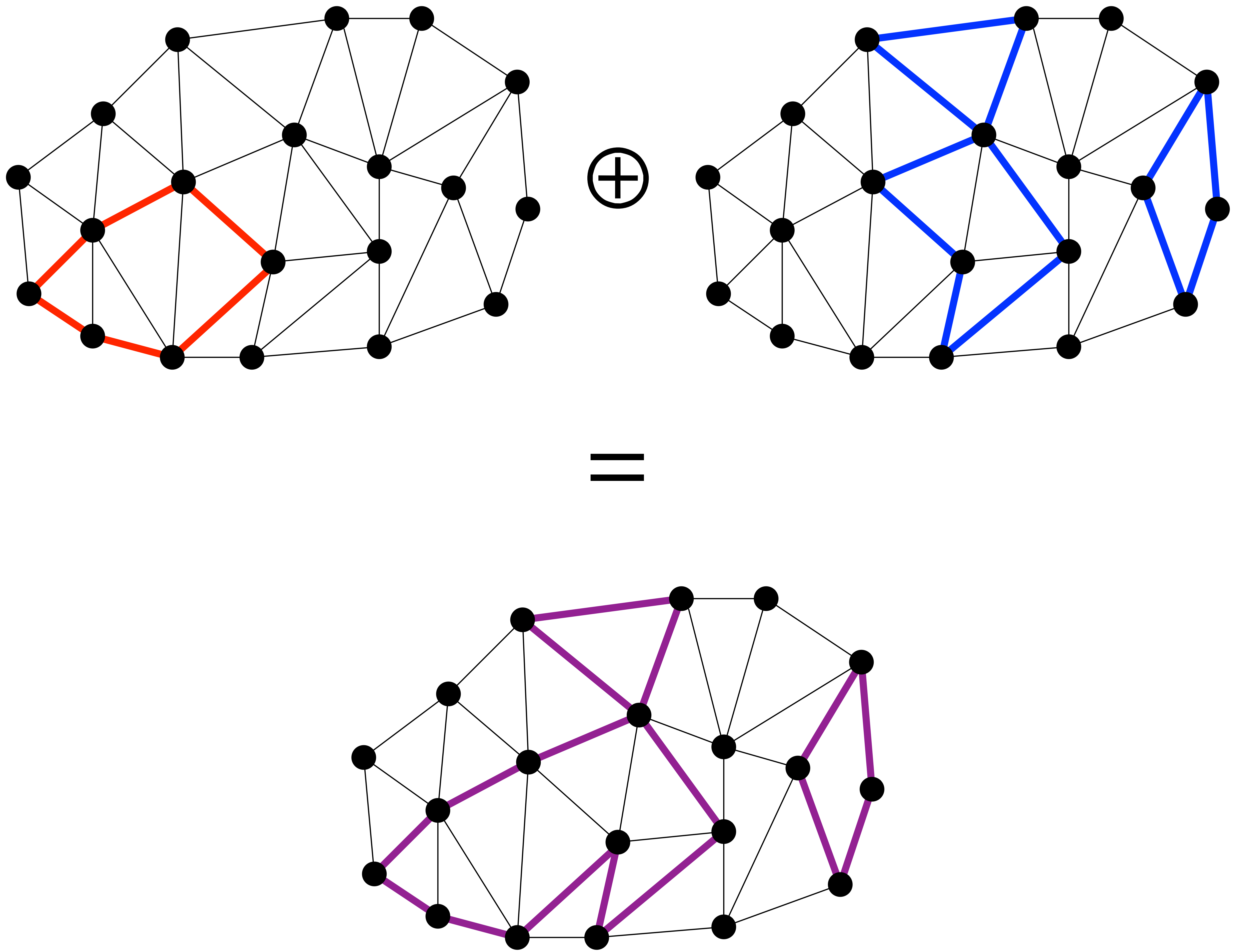


“even-degree subgraph”





Cycle Space

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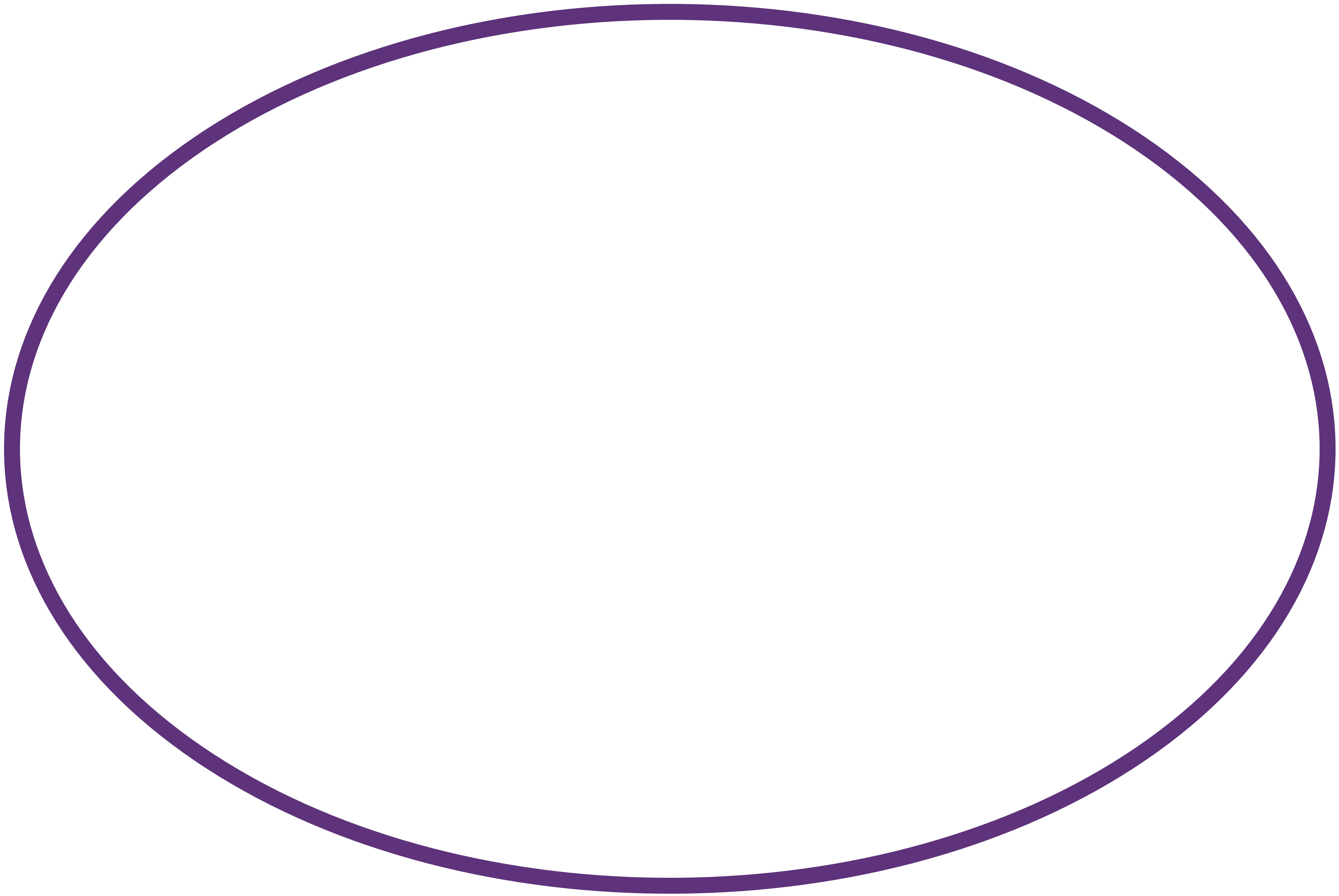
Highlights

How fast?	Who?
$O(m^3 n)$	[Horton '87]
$O(m^3 + mn^2 \log n)$	[de Pina '95]
$O(n^\omega)$ (randomized)	[Amaldi *et al.* '09]
$O(nm^2 / \log n + n^2 m)$	[Mehlhorn, Michail '09]

Planar Graphs



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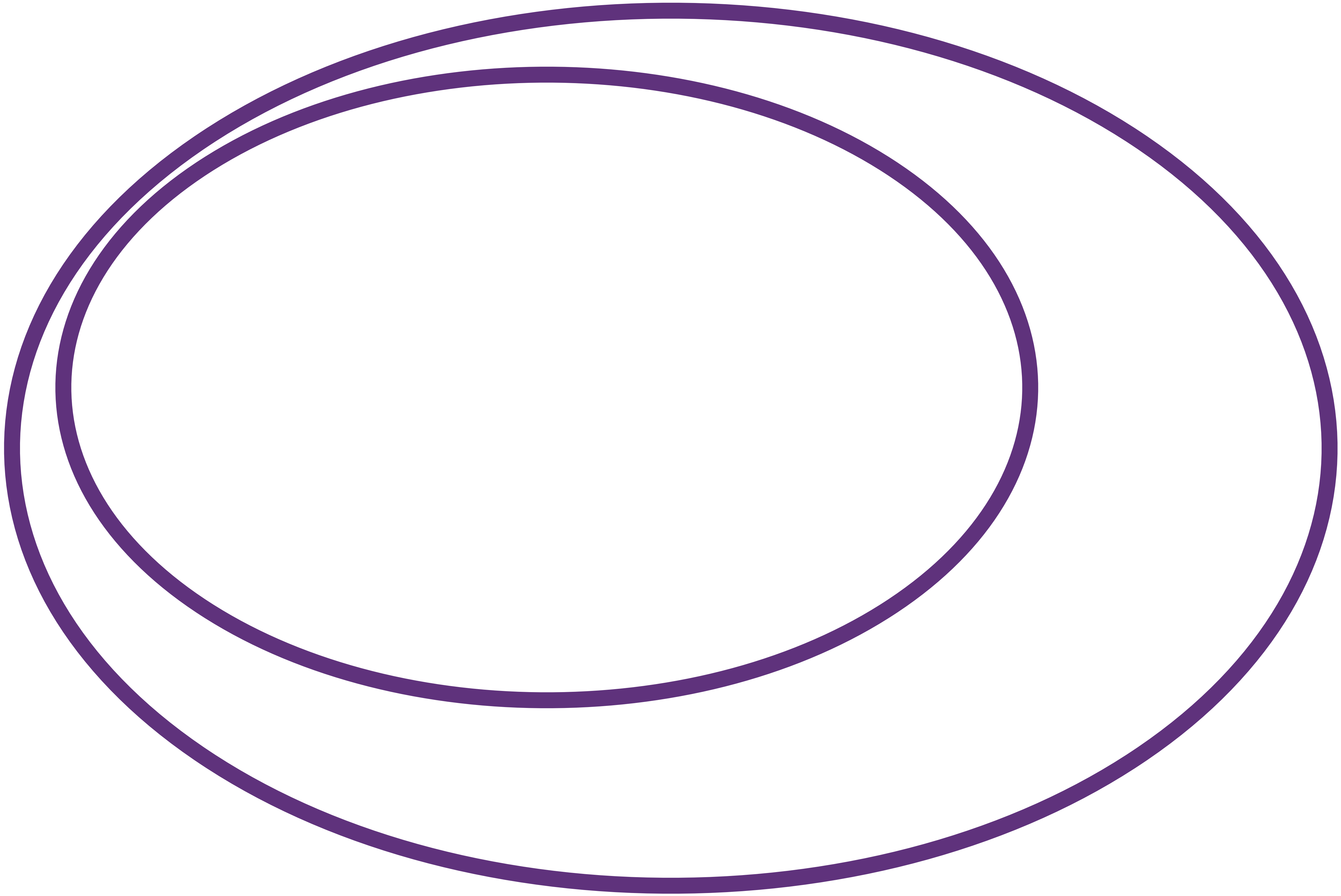

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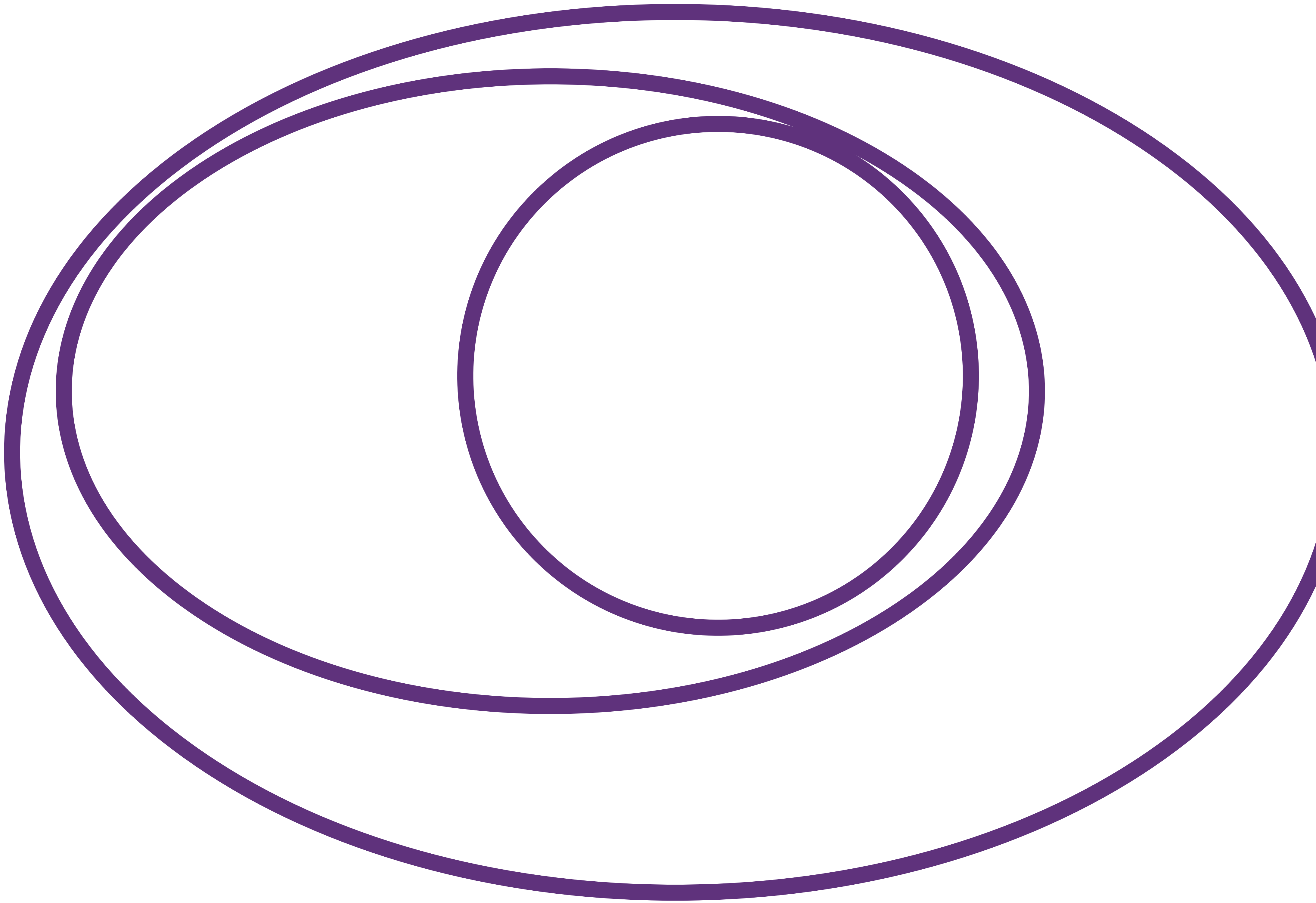

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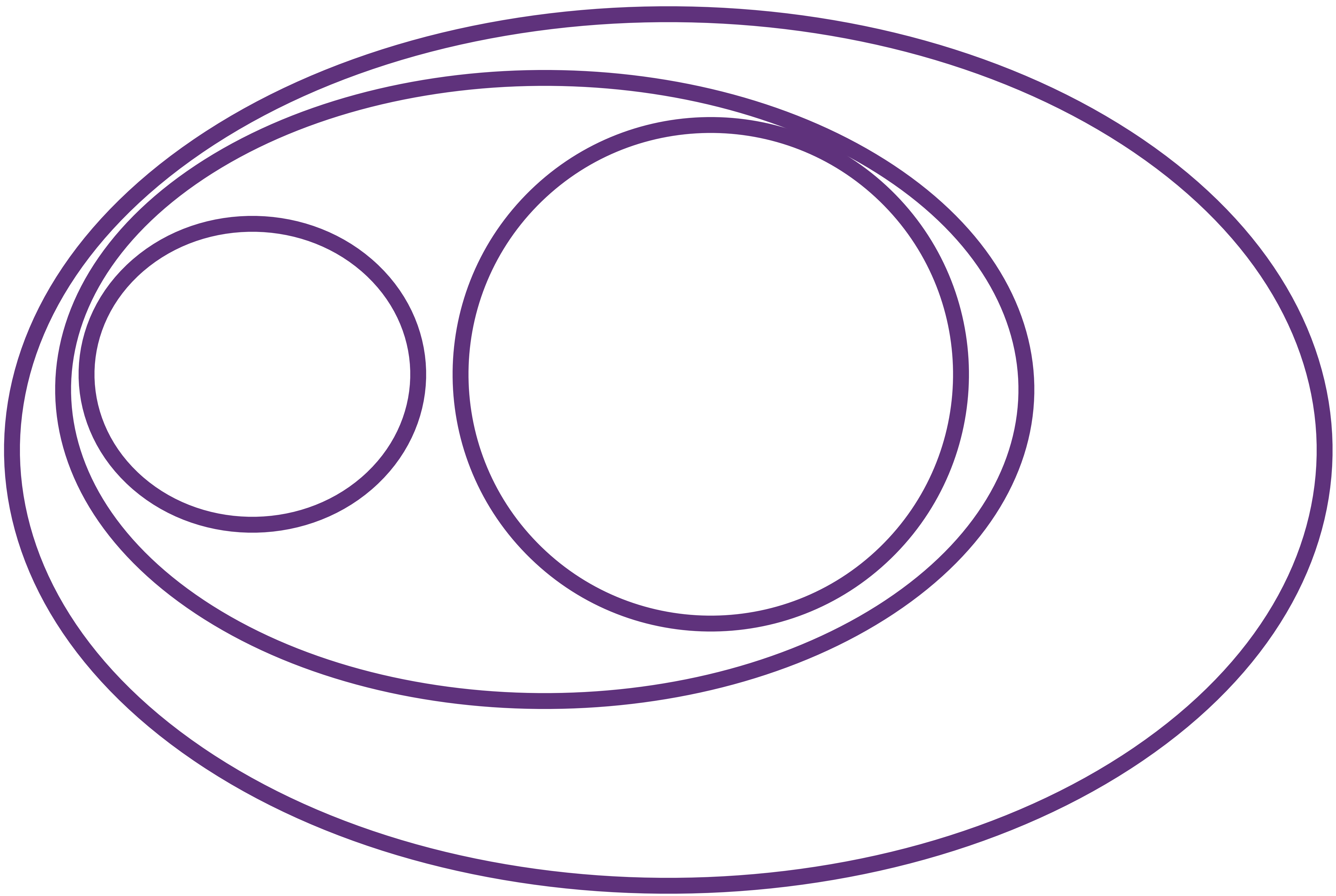

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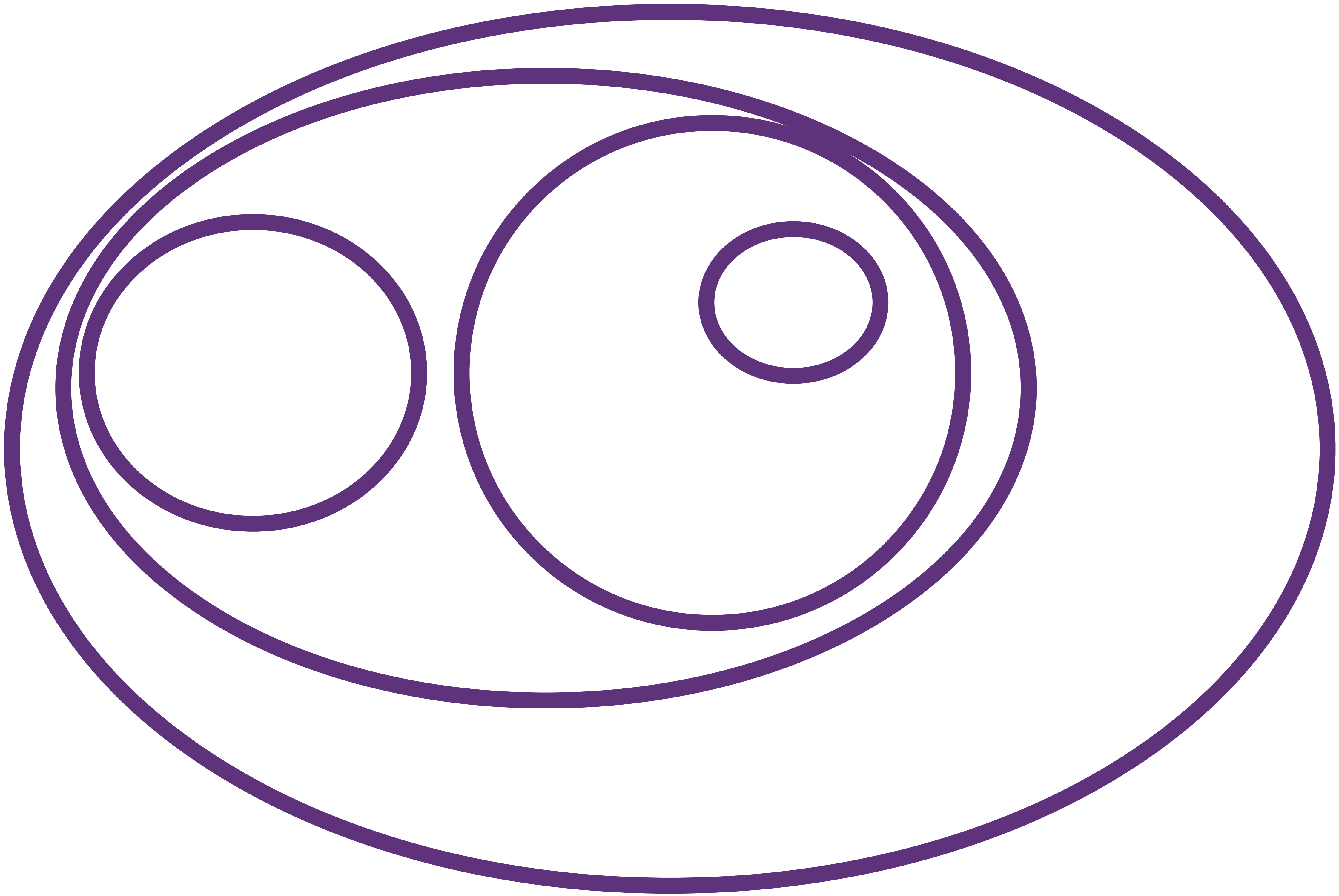

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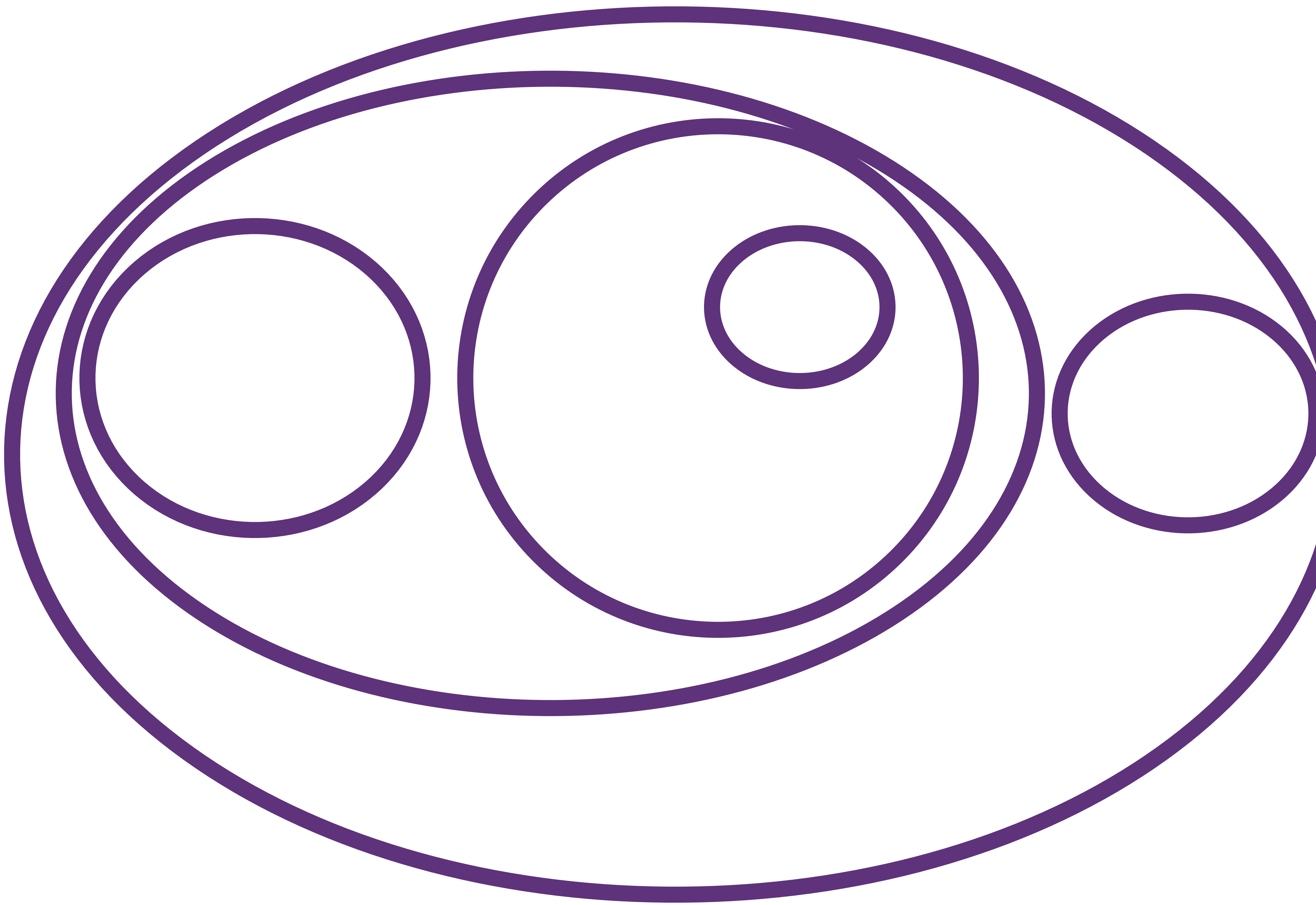

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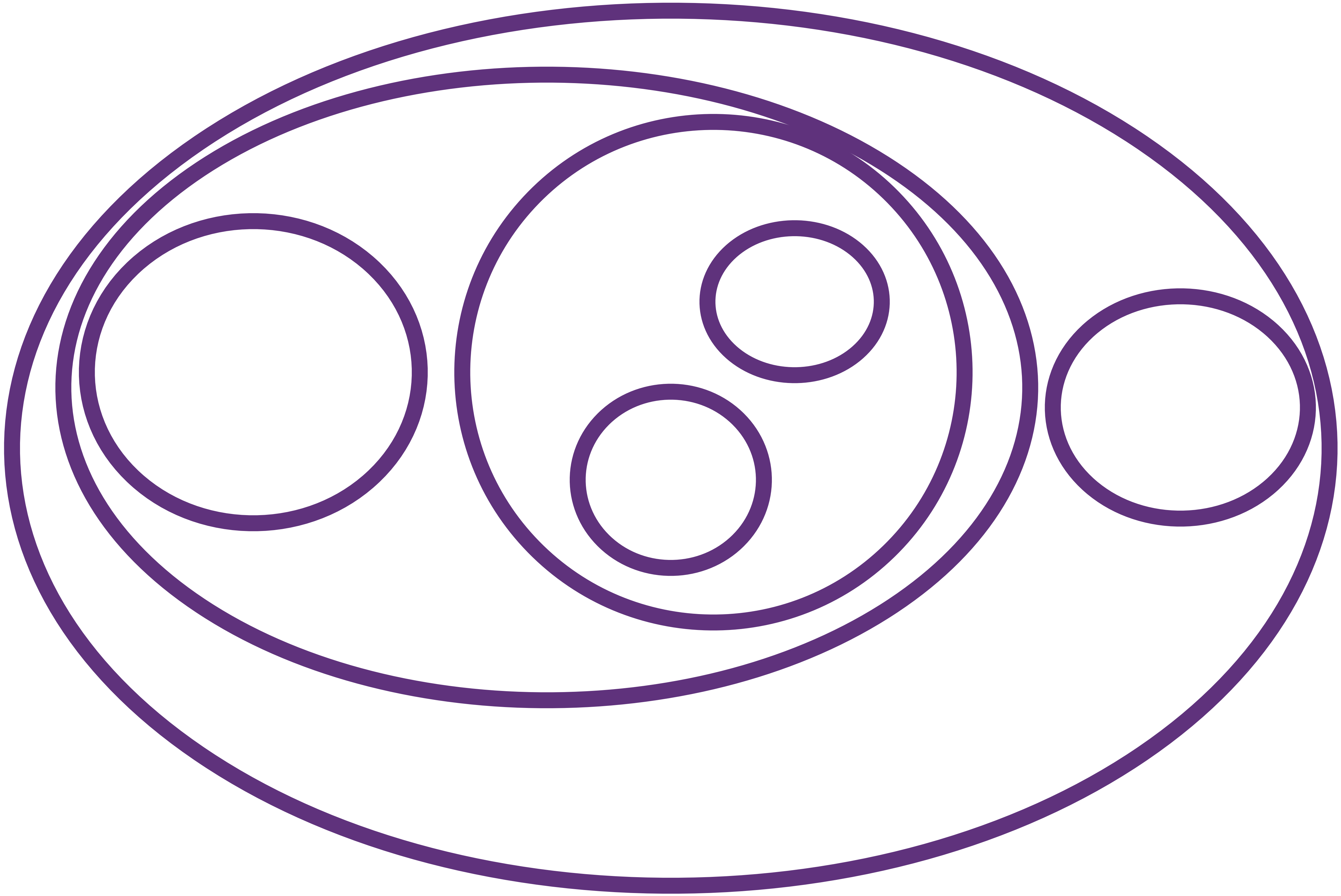

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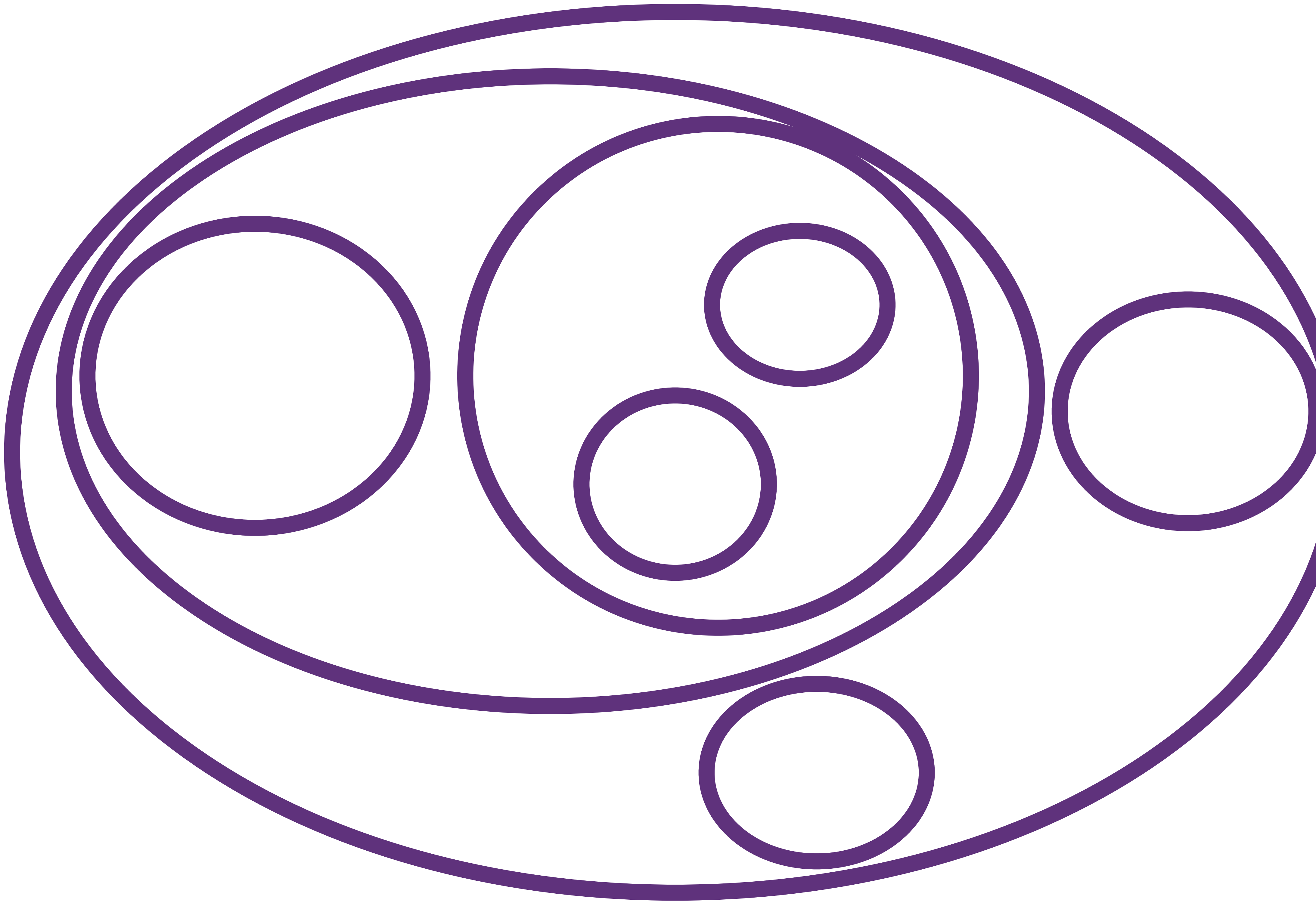

Planar Highlights

How fast?	Who?
$O(n^2 \log n)$	[Hartvigsen, Mardon '94]
$O(n^2)$	[Amaldi *et al.* '09]
$O(n \log^4 n)$ (rand. oracle)	[Borradaile *et al.* '15]
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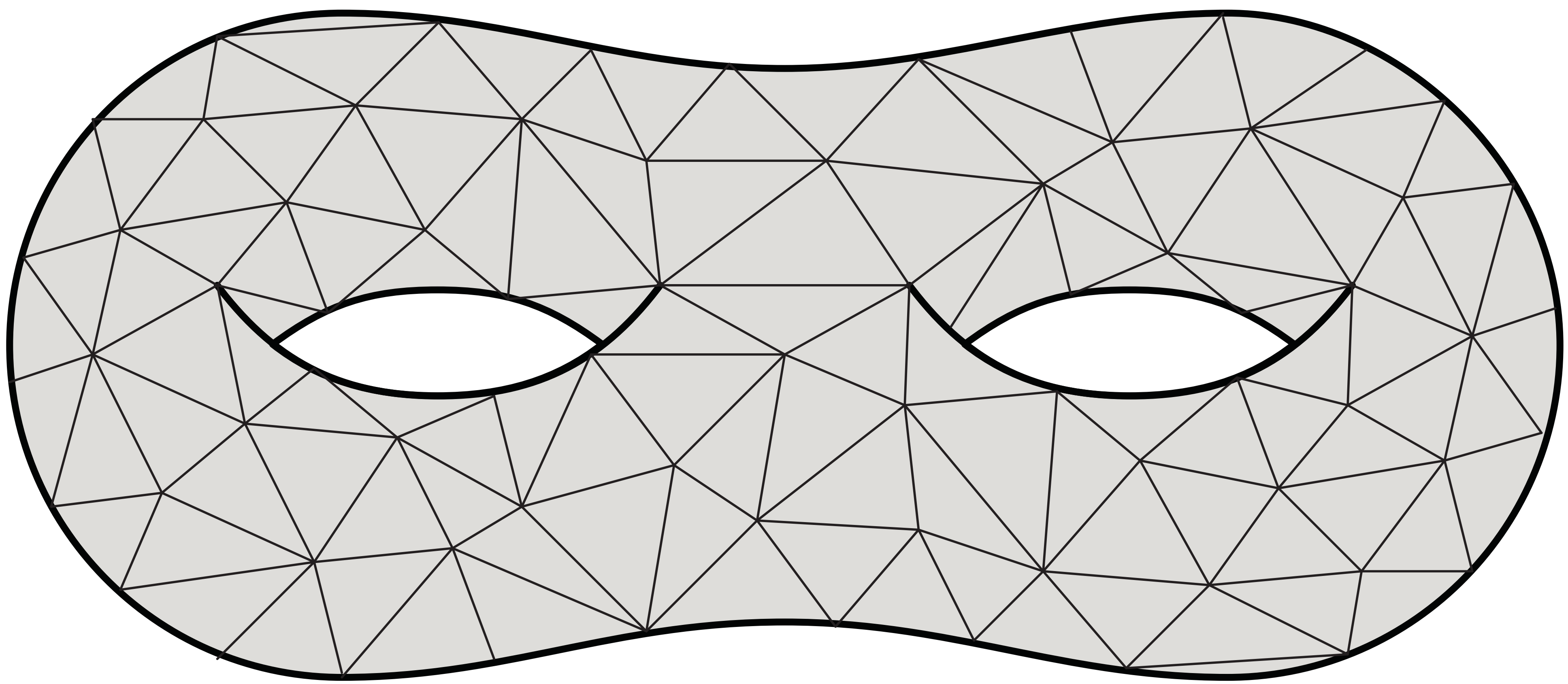
All results also compute minimum cut basis in the dual graph.

Embedded Graphs



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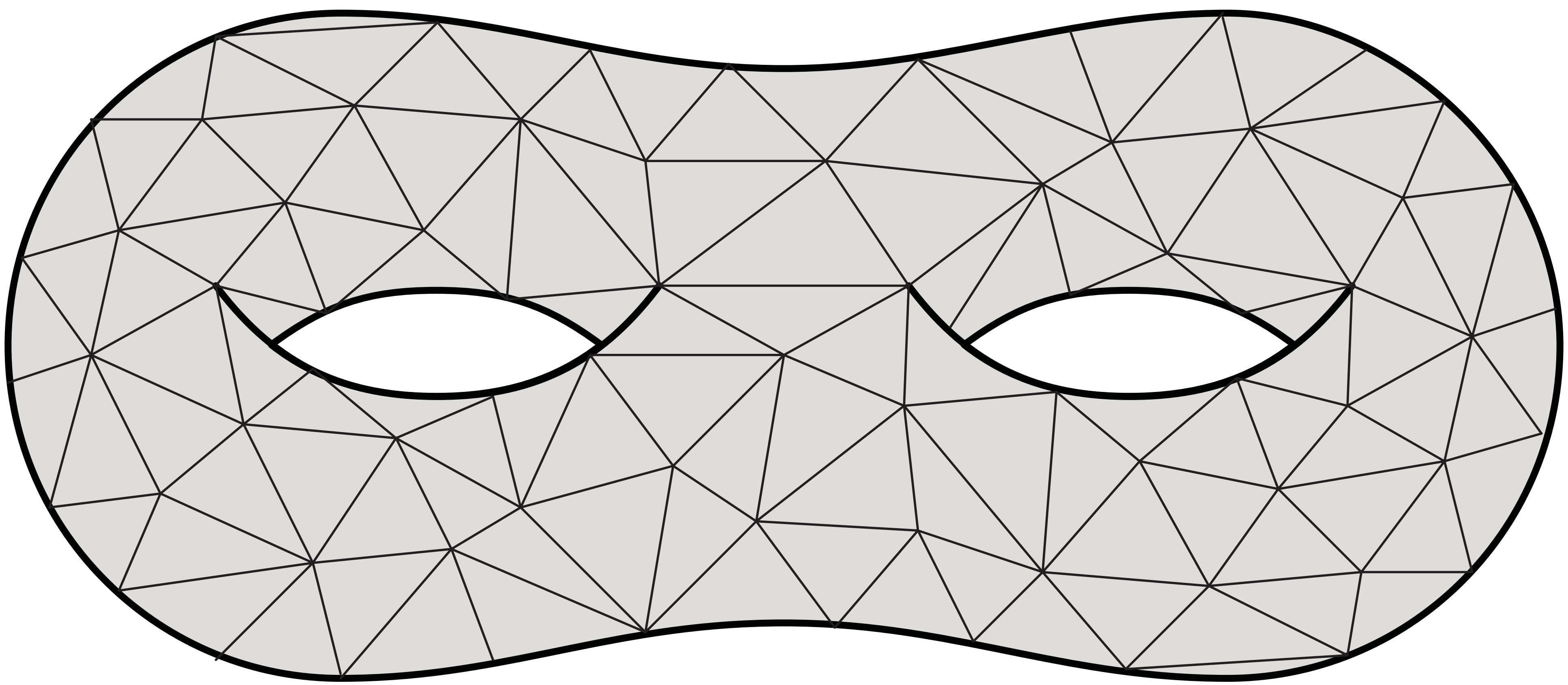

Algorithms often parameterized by genus g

- Minimum *cut* oracle in $2^{O(g^2)} n \log^3 n$ time

  [Borradaile *et al.* (40 minutes ago)]

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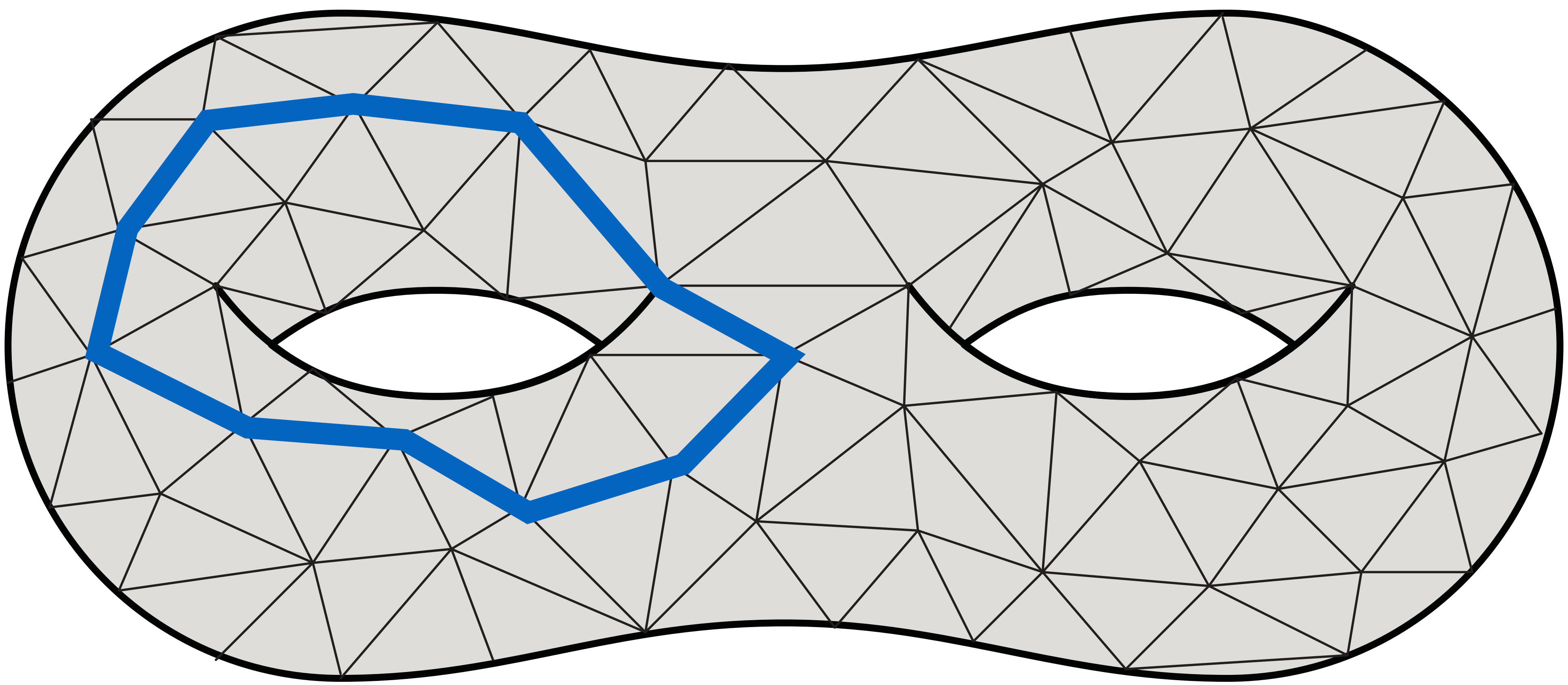
- So there is still have work to do on computing minimum cycle bases

Crossing Cycles



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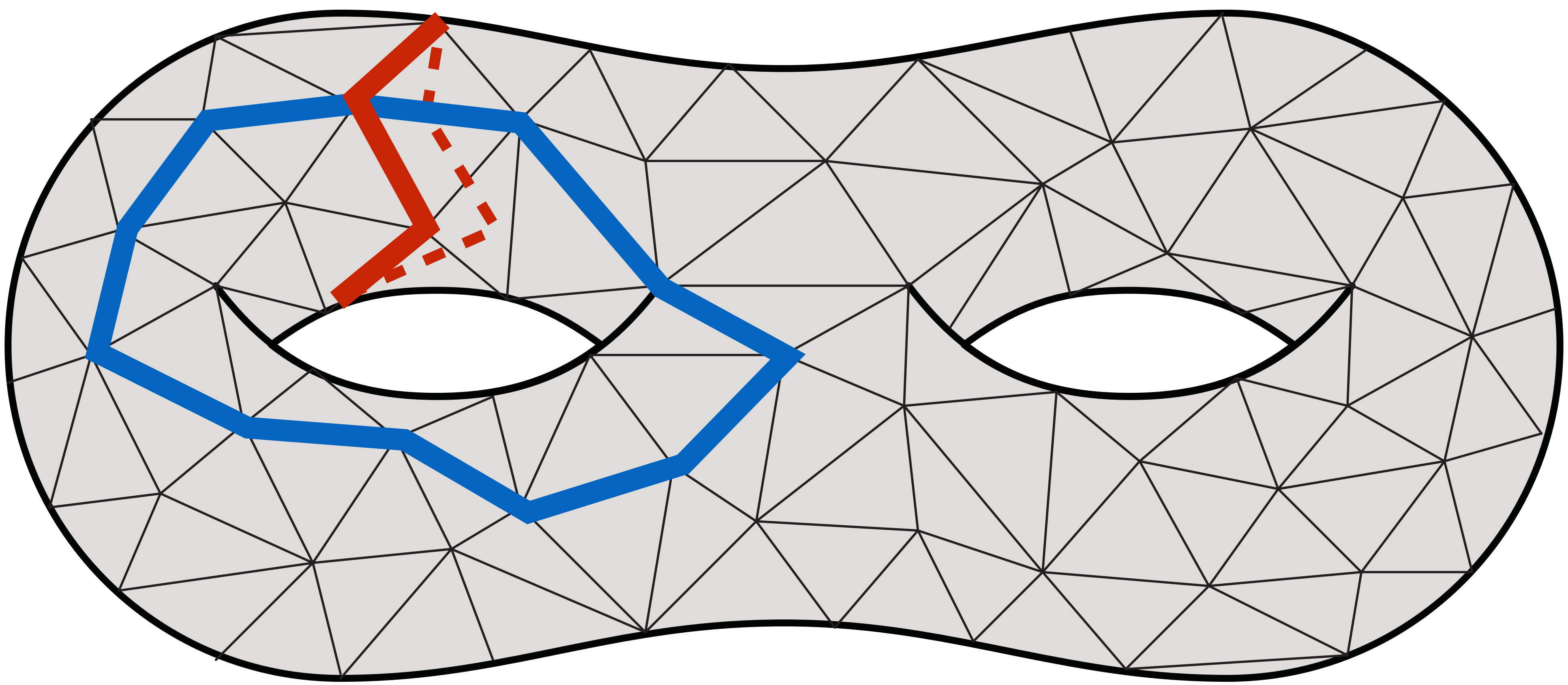
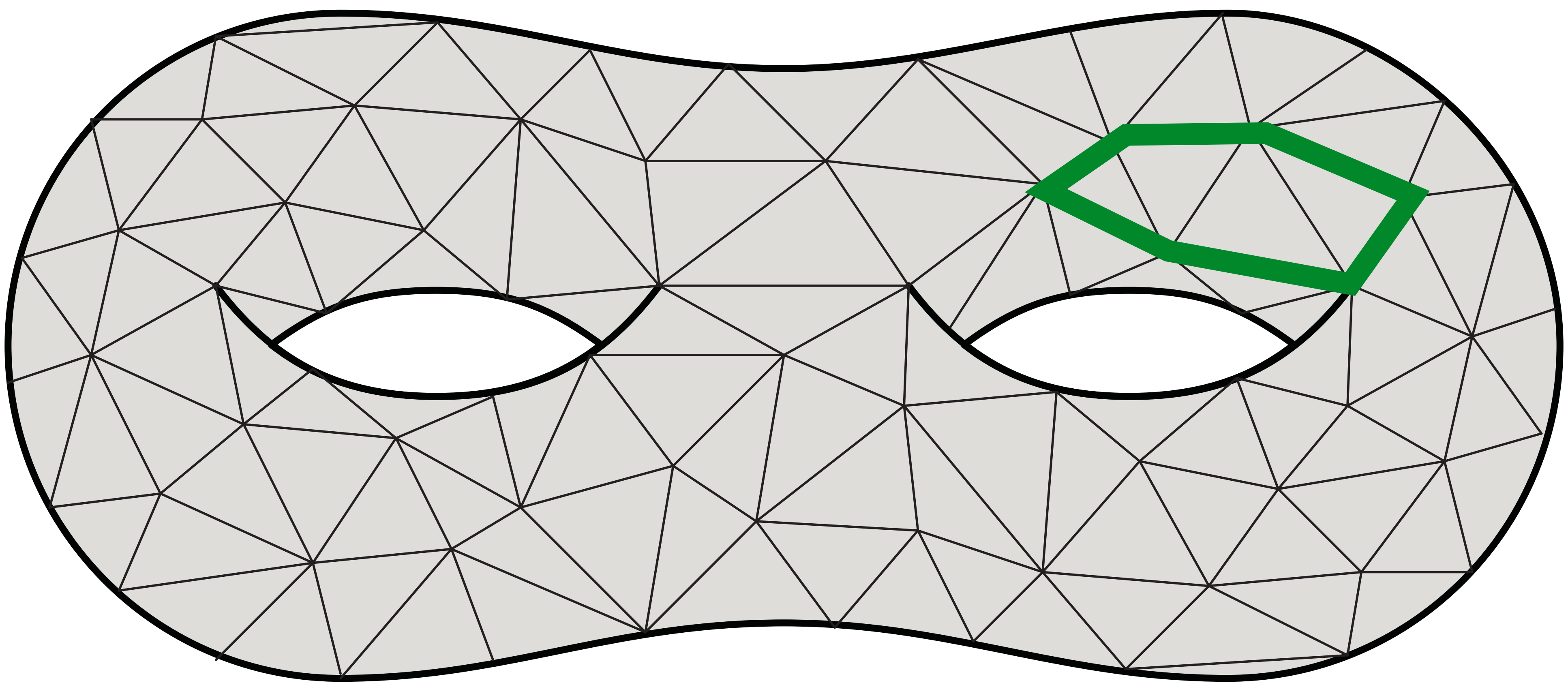


Homology



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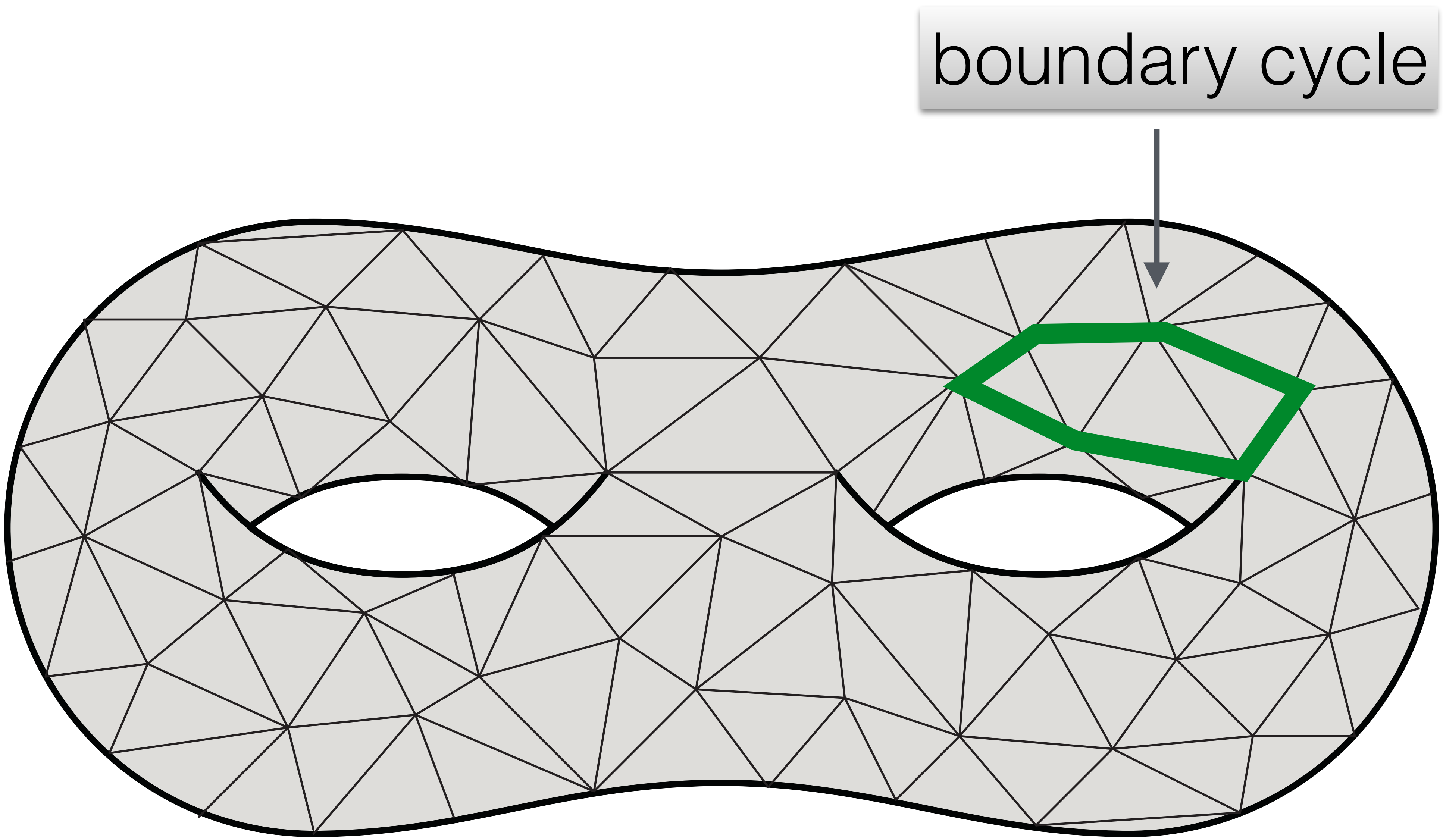

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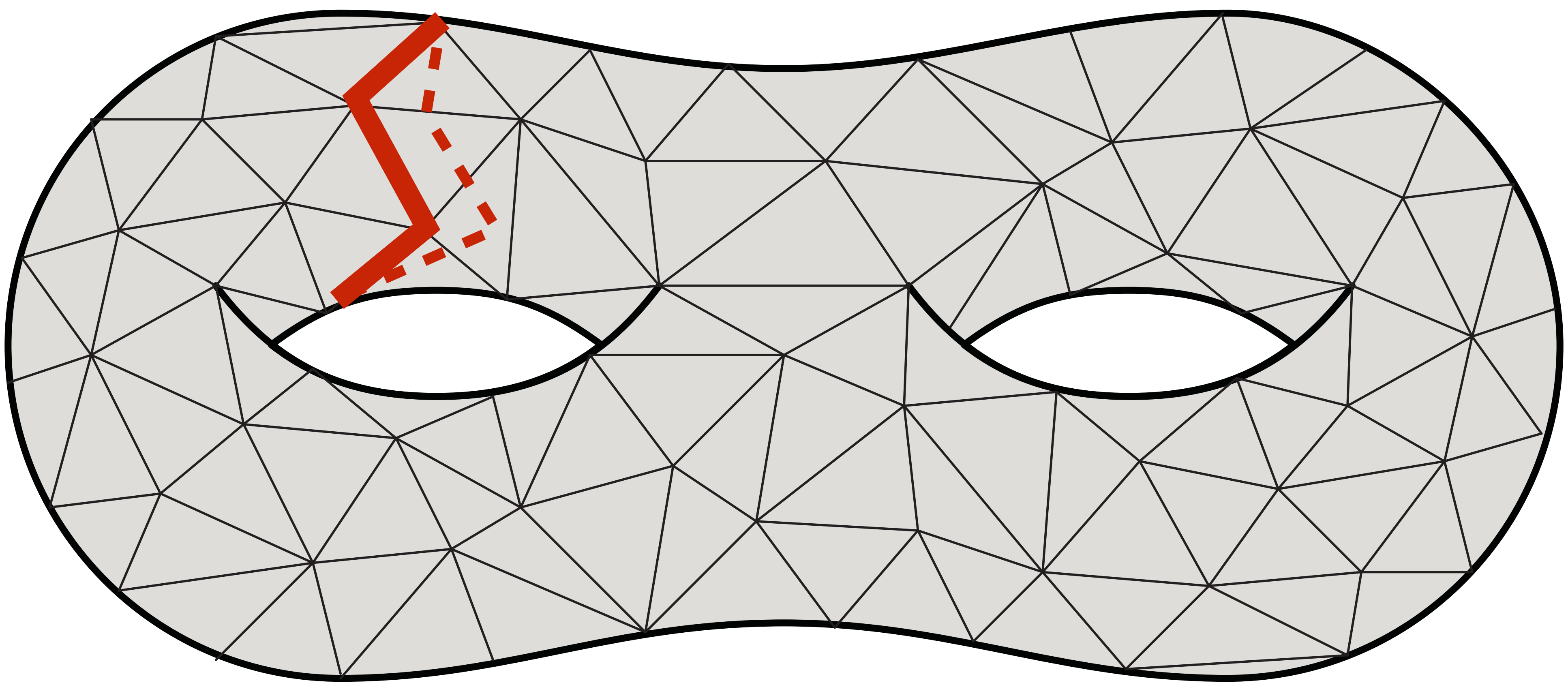

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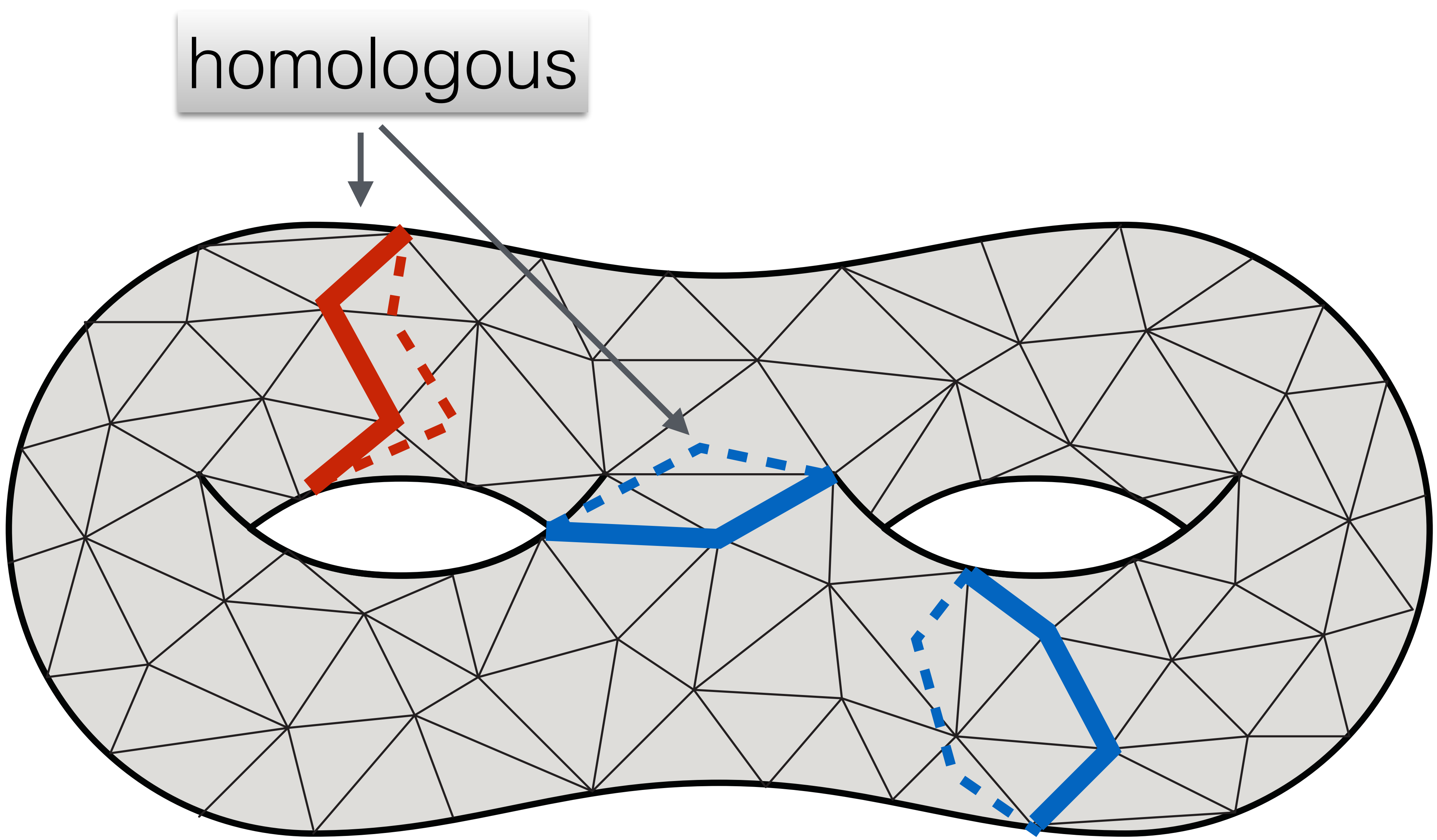

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Cheapest non-boundary cycle in $O(g^2 n \log n)$ [Cabello *et al.* '13]

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- Naively takes $O(n^2)$ time to find γ_j

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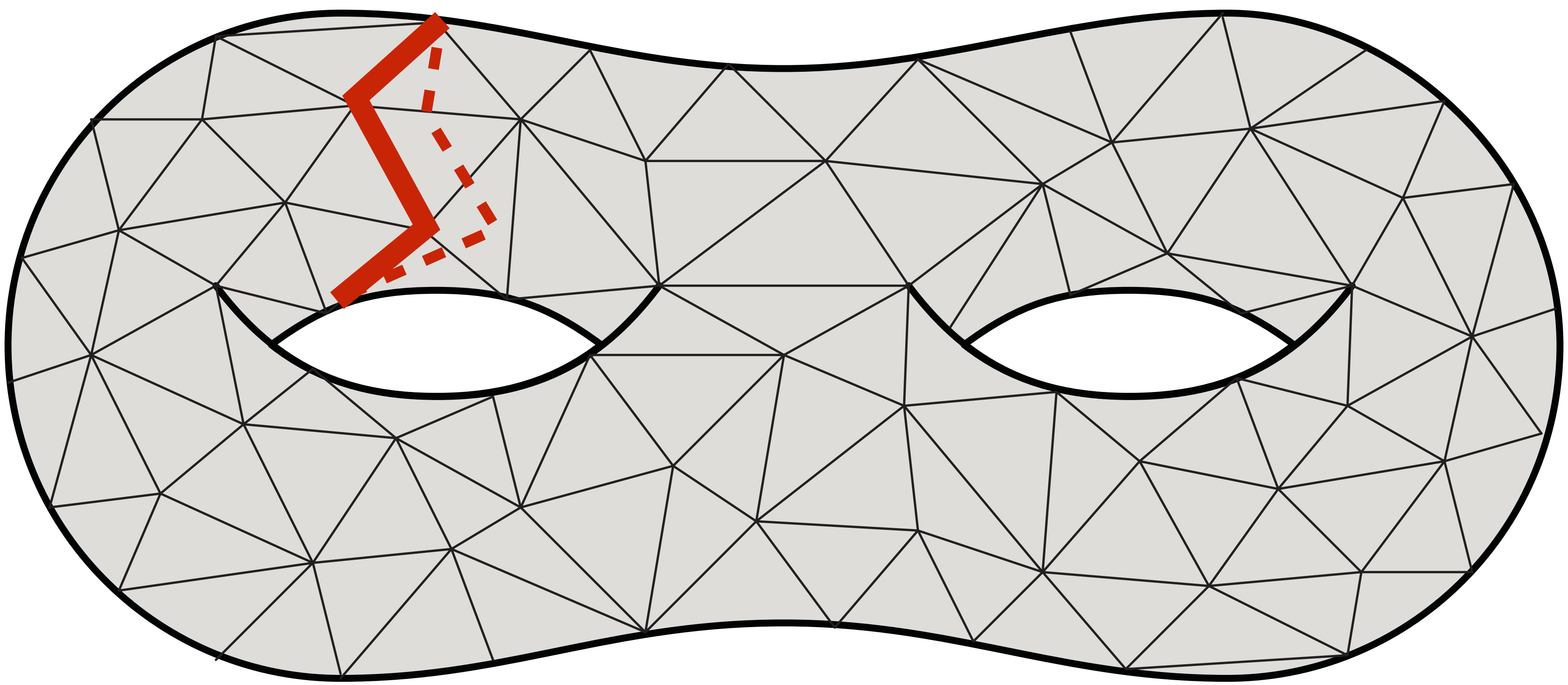
- We consider the complete set of isometric cycles as candidates for each basis cycle γ_j

Homologous Isometric Cycles



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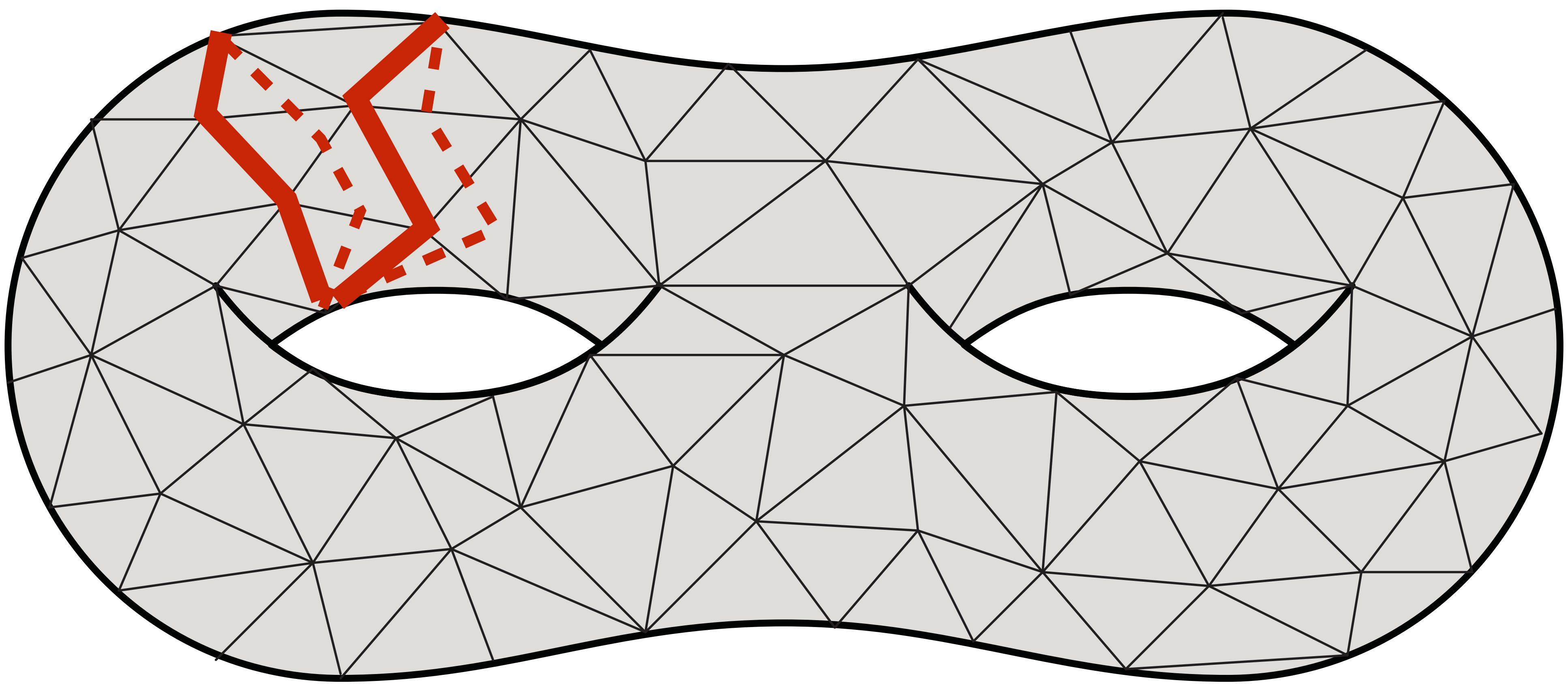

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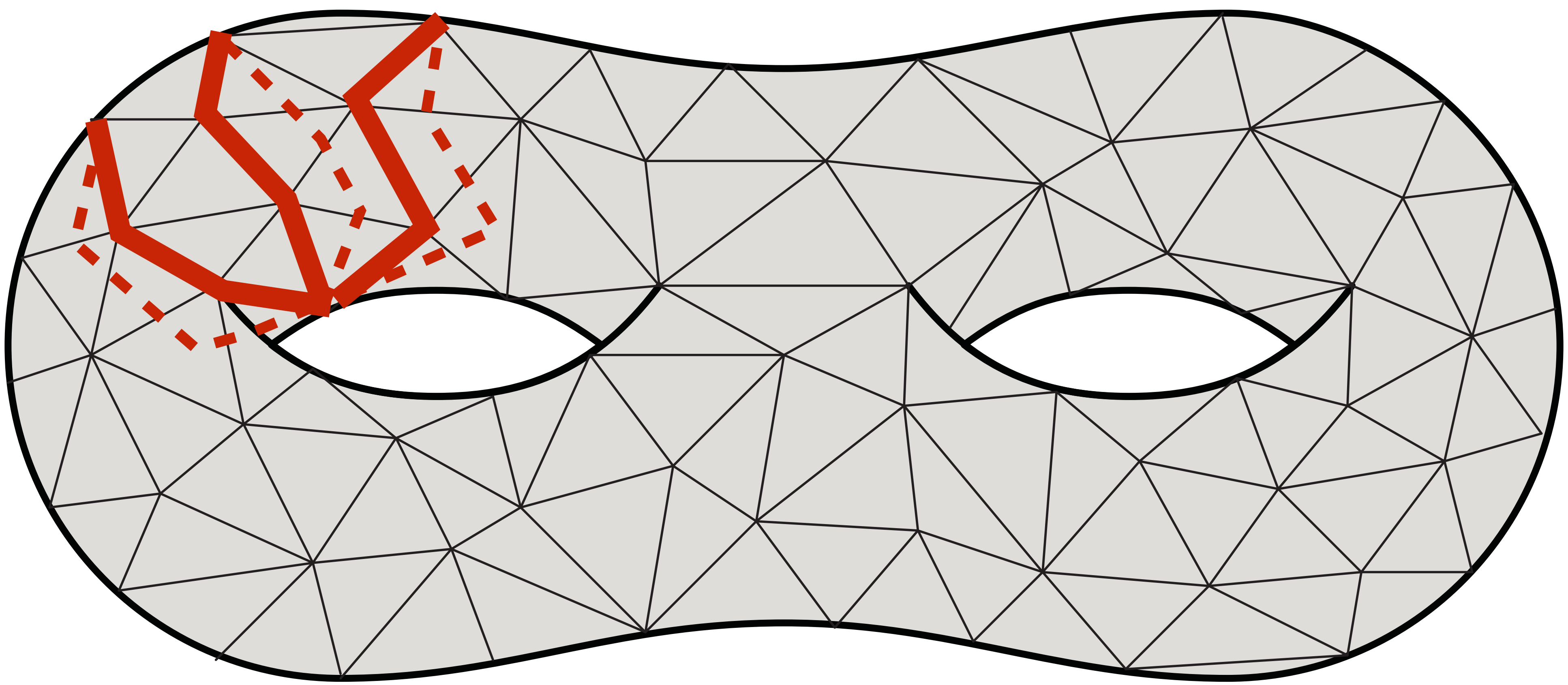

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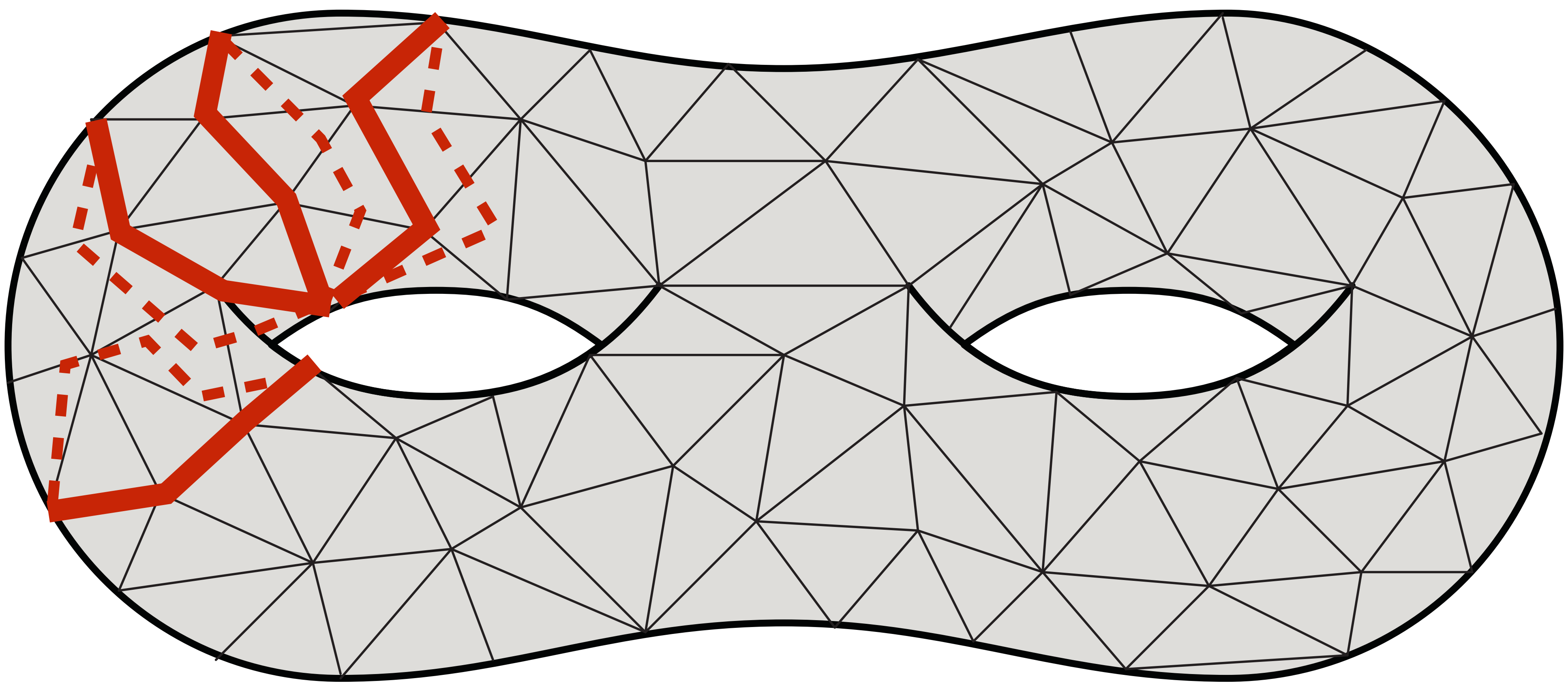

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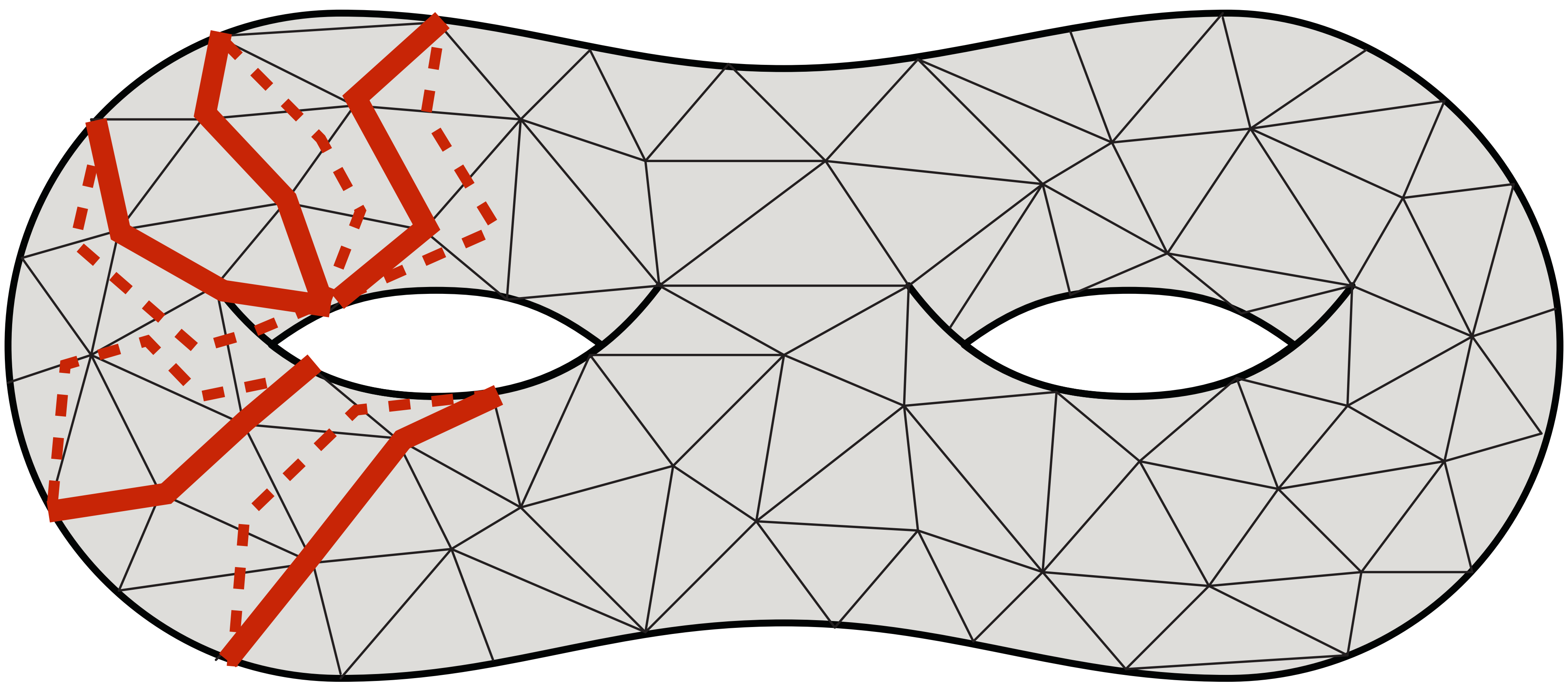

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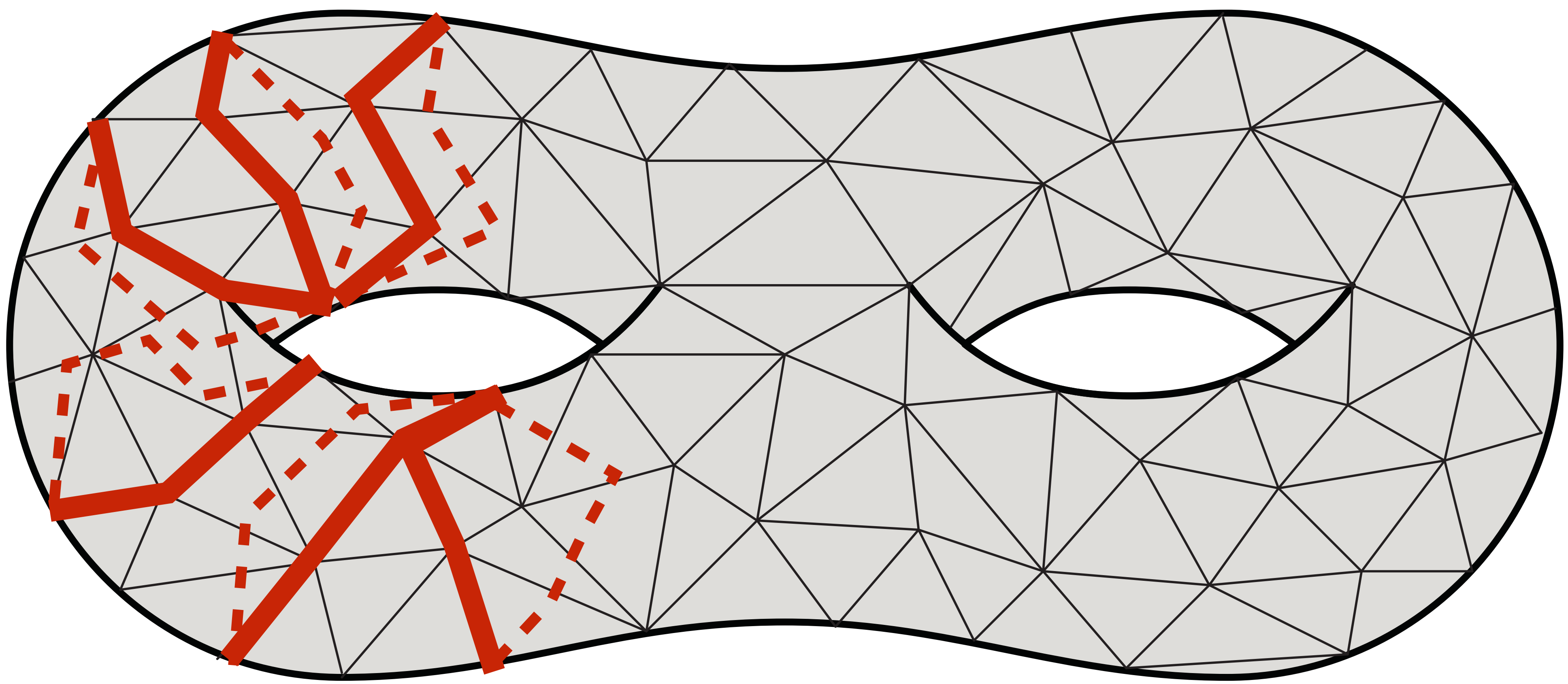

Region Trees

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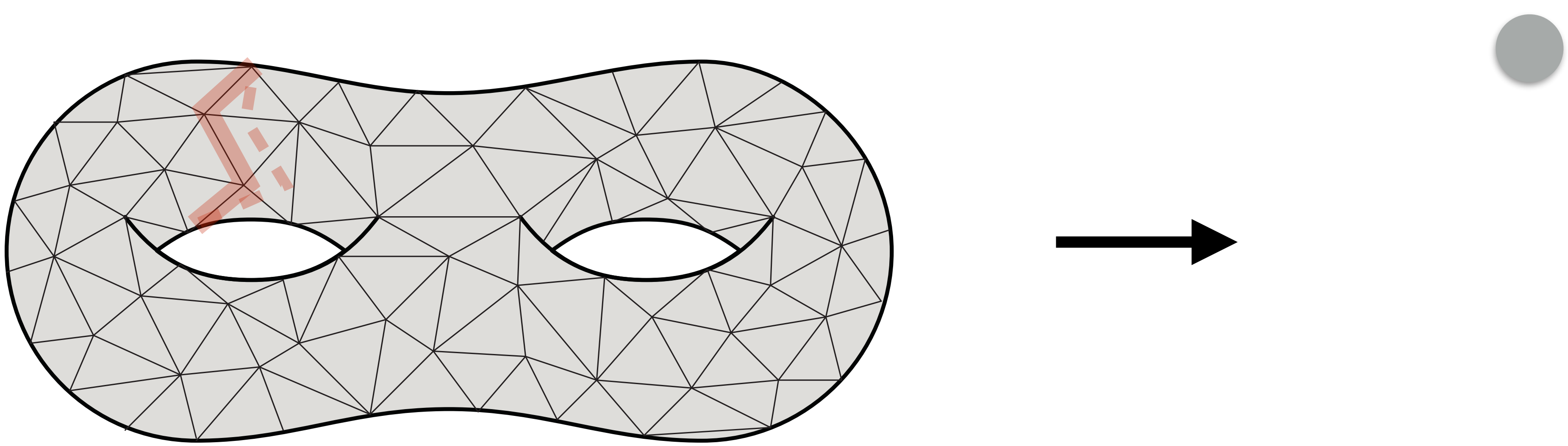

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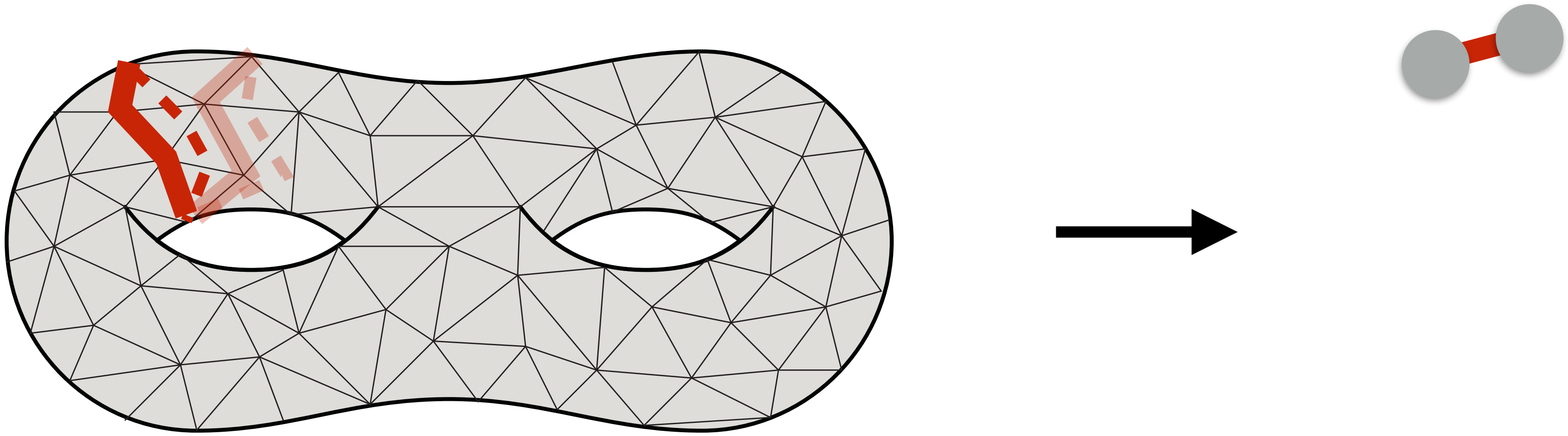

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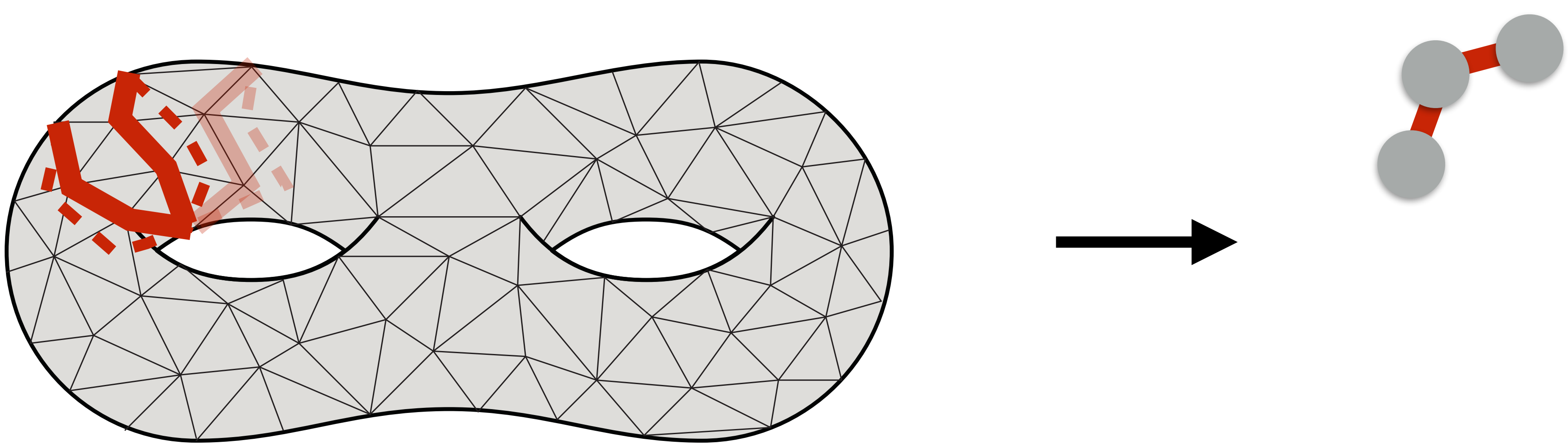

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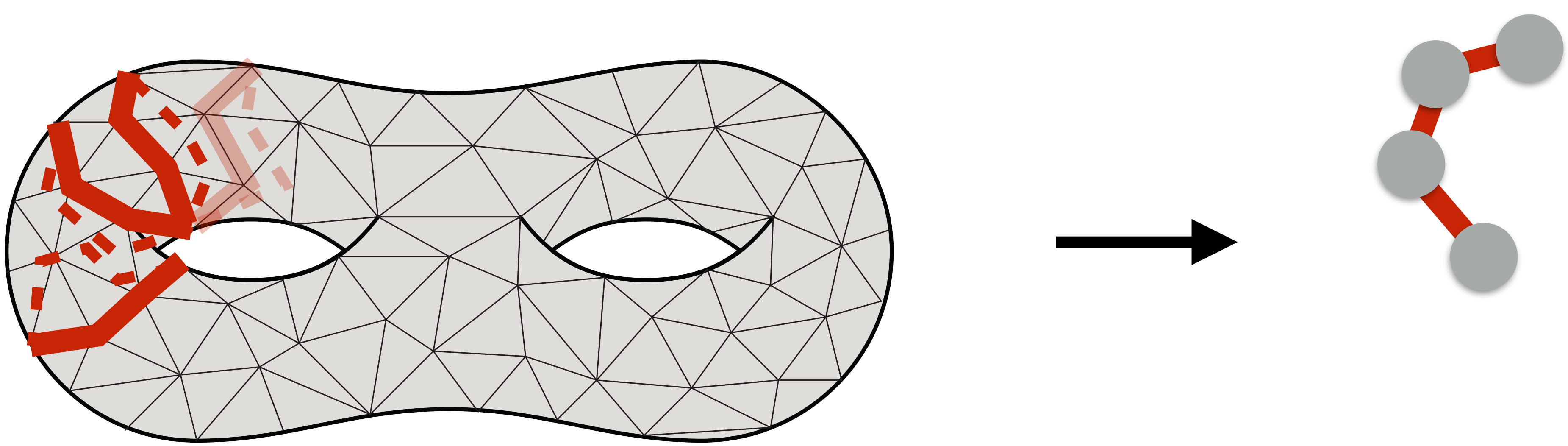

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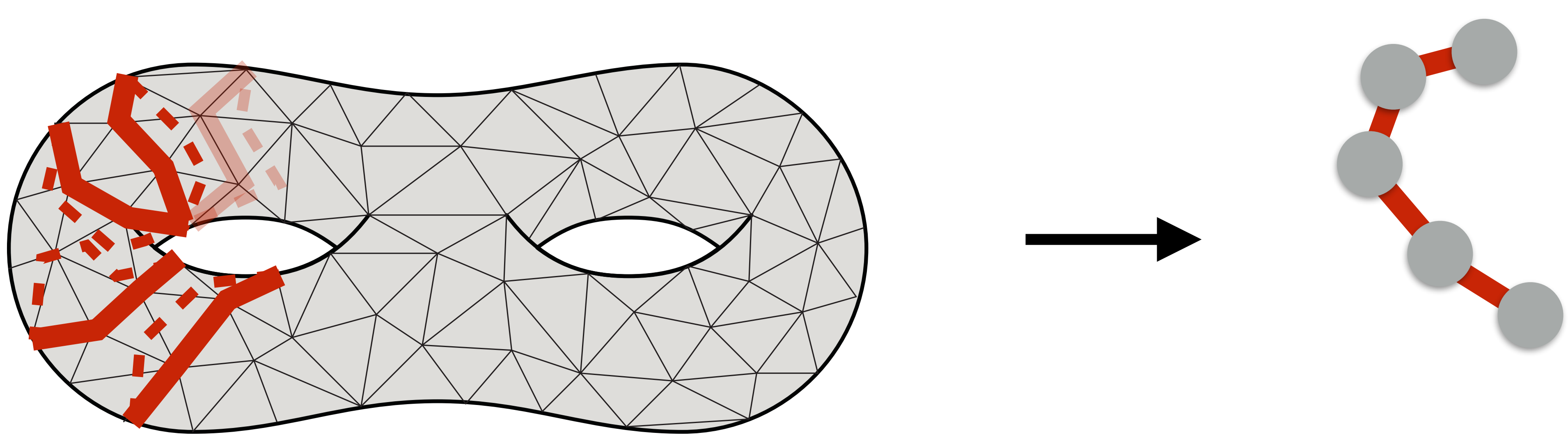

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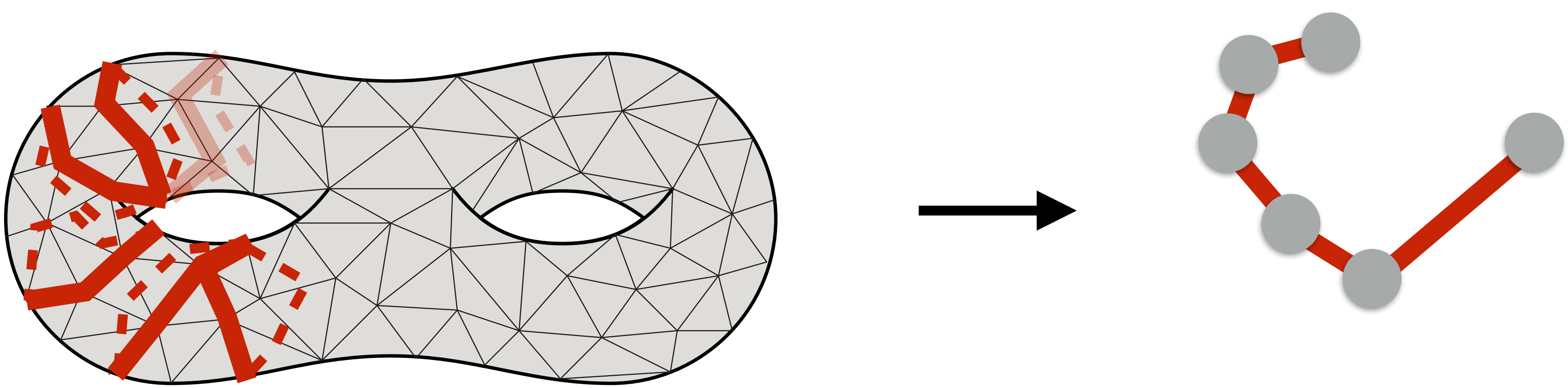

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- Overall, $O(2^{2g} n^2)$ time spent picking basis cycles

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- Reducing the time spend maintaining $S_1, \ldots, S_{m-n+1}$ means reducing the overall running time!

Homology Basis

- Minimum homology basis algorithm uses de Pina's algorithm with $2g$-bit signatures instead

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- Searches extension of cyclic double cover [Erickson '11] to find basis cycles in $O(g^2 n \log n)$ time each

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