Global Minimum Cuts in Surface Embedded Graphs

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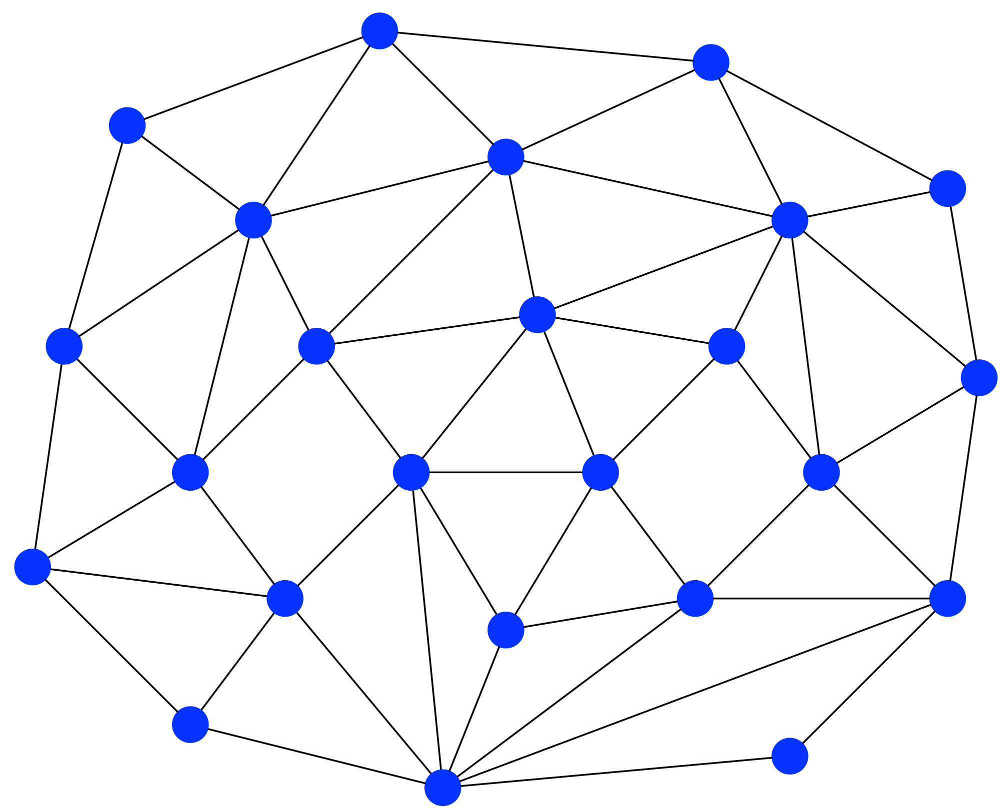
Minimum Cut

- Partition vertices of an undirected edge-weighted graph into two non-empty sets
- Minimize weight of edges between sets



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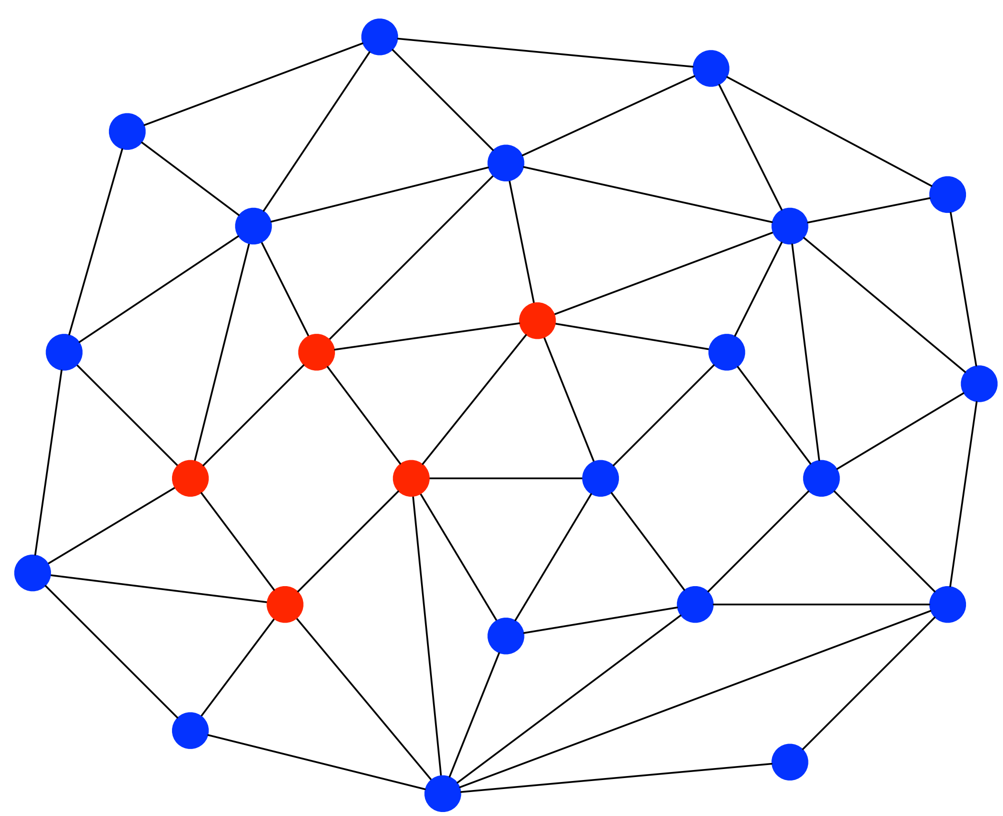

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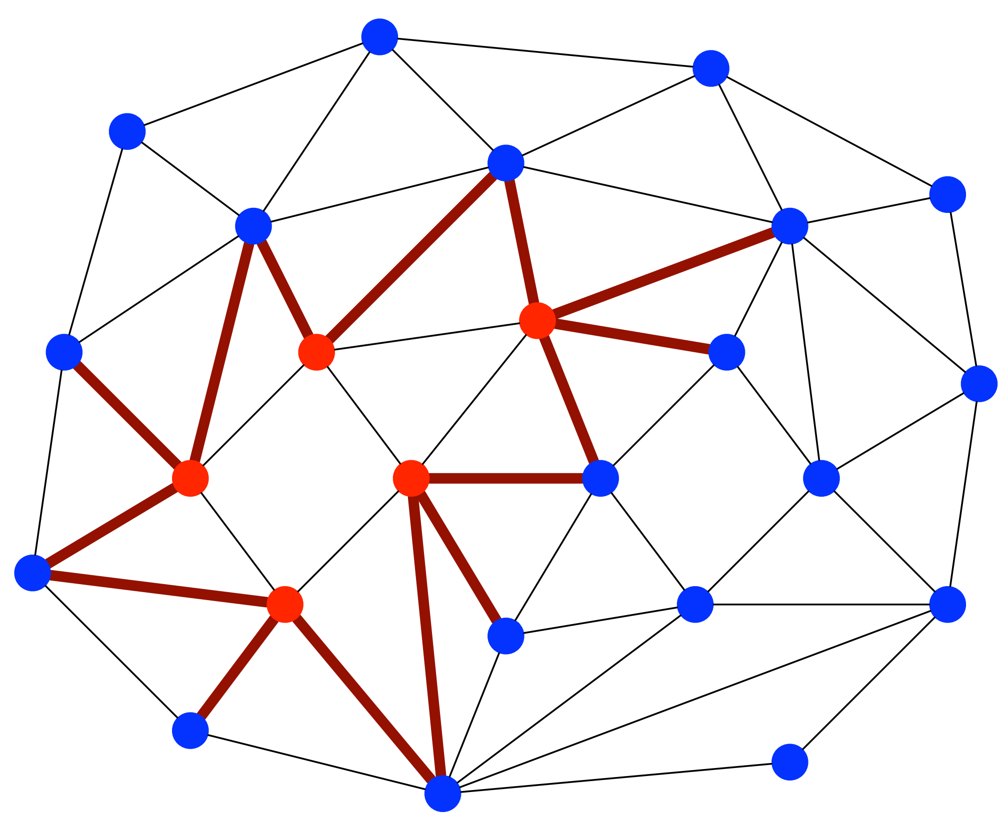


Minimum Cuts

- $O(nm + n^2 \log n)$ deterministic [Nagamochi, Ibaraki '92]
- $O(m \log^3 n)$ randomized [Karger '00]

Planar Cuts

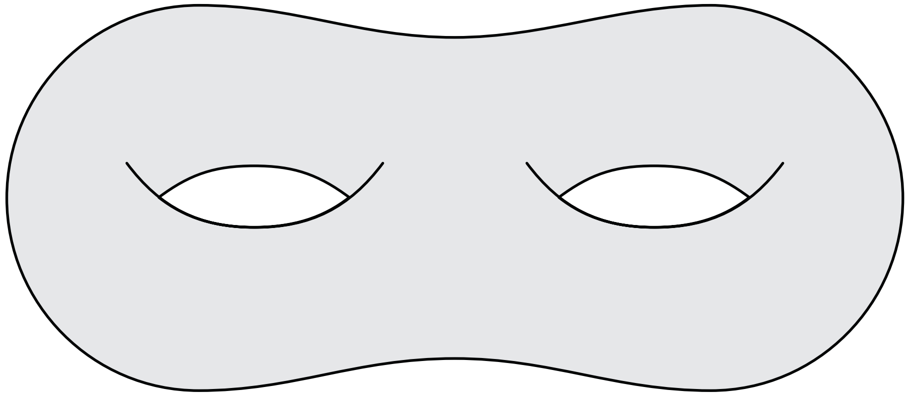
- $O(n^2 \log n)$ deterministic [N, I '92]
- $O(n \log^2 n)$ randomized [K '00]
- $O(n \log^2 n)$ deterministic [Chalermsook, Fakcharoenphol, Nanongkai '04]
- $O(n \log n \log \log n)$ deterministic [Italiano *et al.* '11]
- $O(n \log \log n)$ deterministic [Łącki and Sankowski '11]

Planar Graphs

- Supports thesis of "planar = fast"
- Thesis supported by *s*,*t*-cuts, maximum flows, shortest paths, minimum spanning trees, graph isomorphism, approximate TSP, approximate Steiner tree, …
- Numerous generalizations of planar graphs

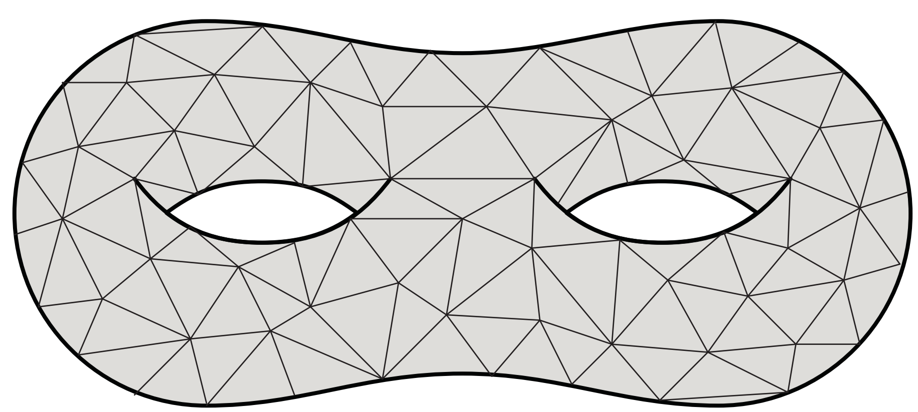
Surfaces

- 2-manifolds (with boundary)
- ***genus g***: max # of disjoint simple cycles whose compliment is connected
  = number of holes
  = number of handles attached to sphere



Surface Graphs

- Generalizes planar graphs
- *Most* planar results generalize easily
- *s*,*t*-cuts and flows only recently [Chambers, Erickson, Nayyeri STOC/SOCG '09; Italiano et al. '11]



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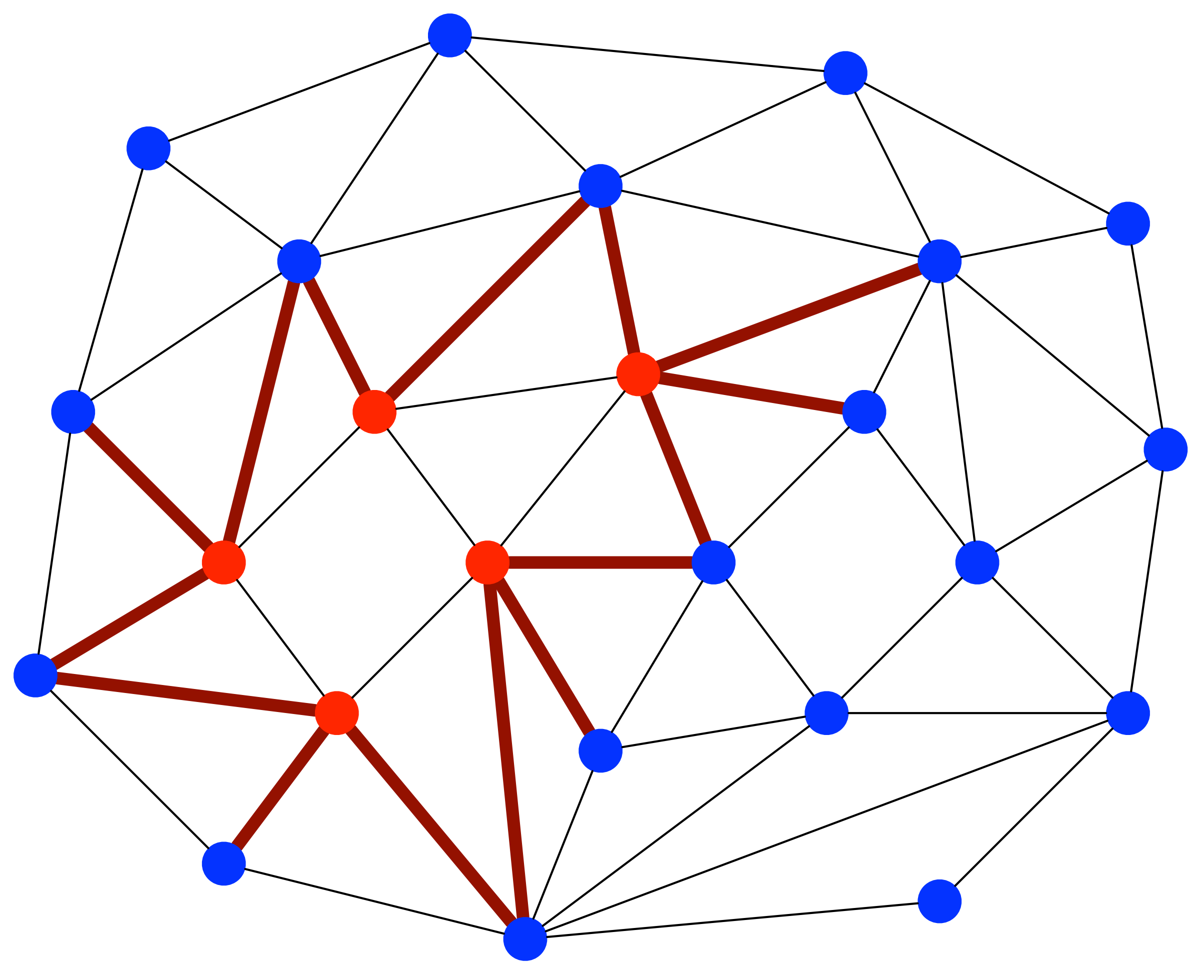
- Matches time bound of Łącki and Sankowski for planar graphs
- Also matches time bound for minimum s,t-cuts by Italiano *et al.*

Back to the Plane

- Minimum cut is dual to the minimum cycle



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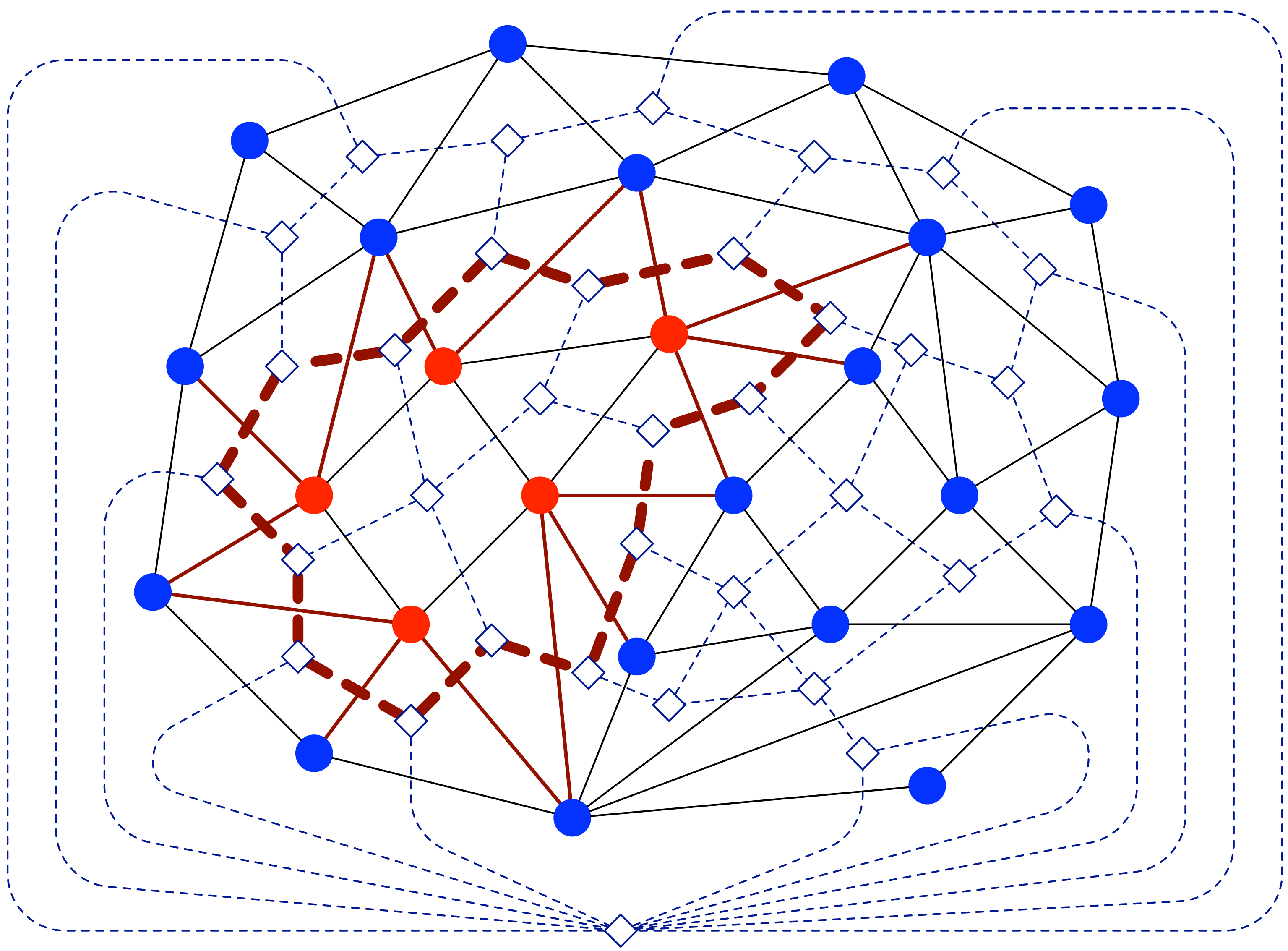
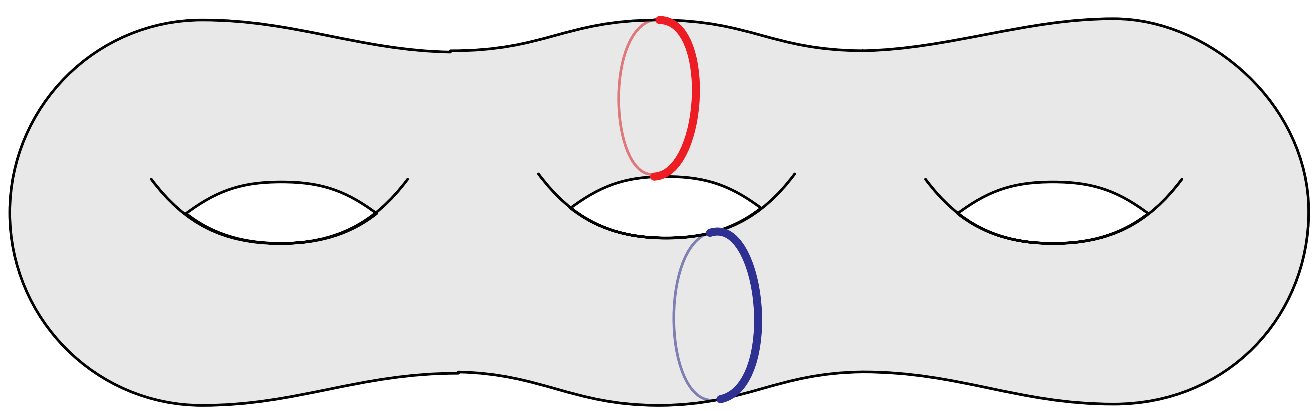


Difficult to Generalize

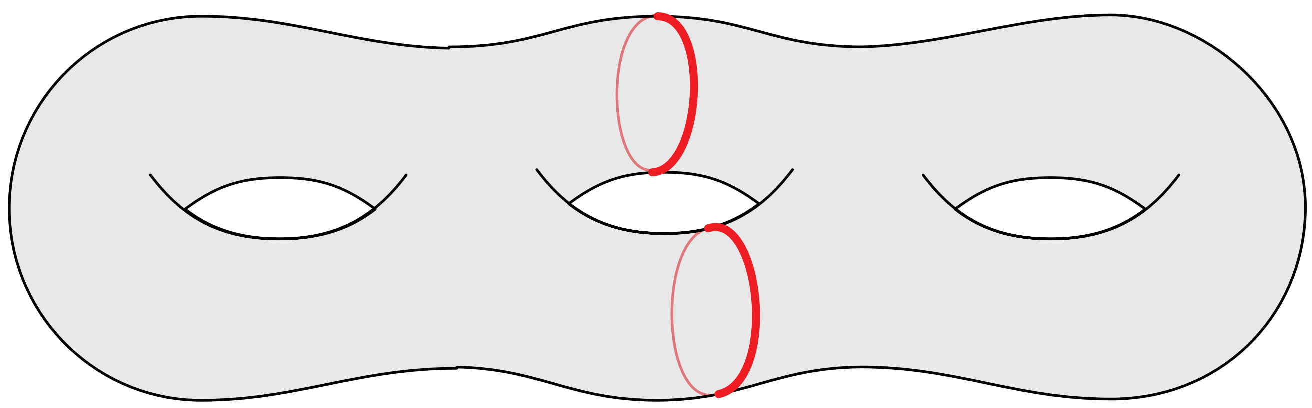
- Cycles may not separate dual faces
- Minimum cut may have multiple components



The Algorithm

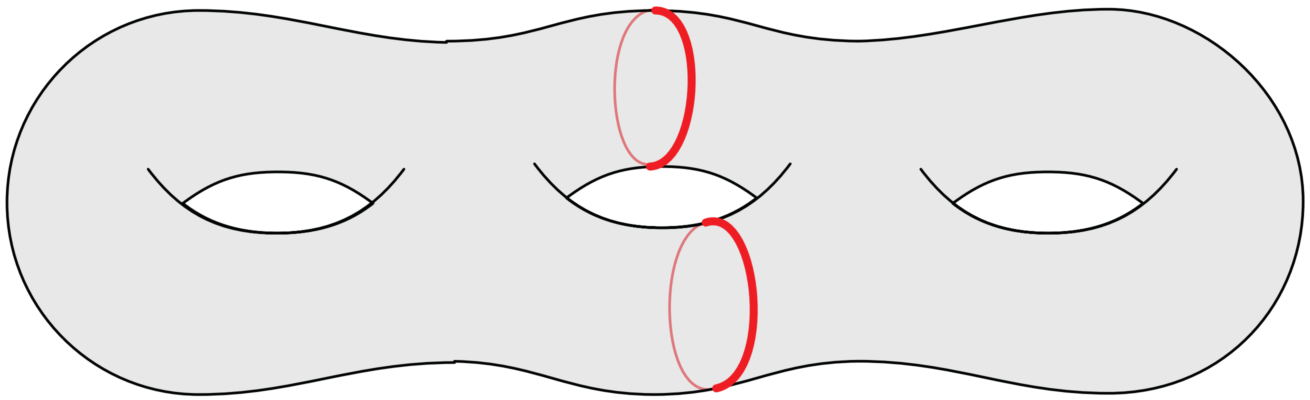
- Is great, but first we need some definitions

Homology



- An ***even subgraph*** has even degree on each node
- An even subgraph is ***separating*** if removing it from the surface disconnects the surface
- Separating subgraphs form the boundary of a subset of faces

Homology

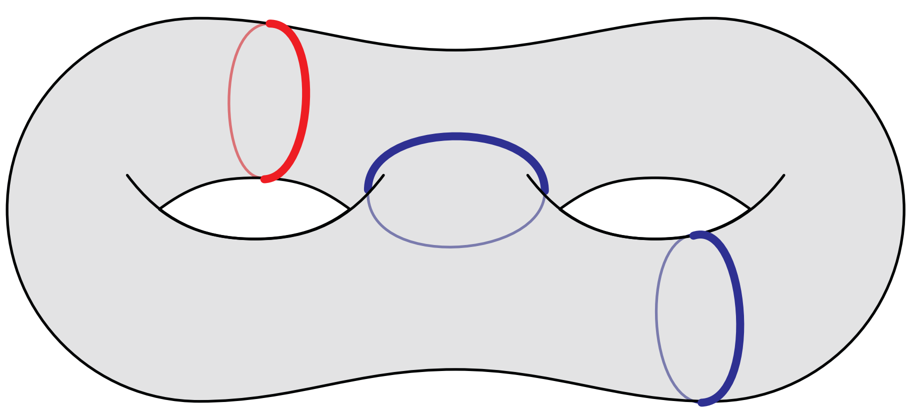


✳ The minimum cut is *dual* to the minimum separating subgraph

- Analogous to main lemma of [Chambers *et al.* '09]

More Homology

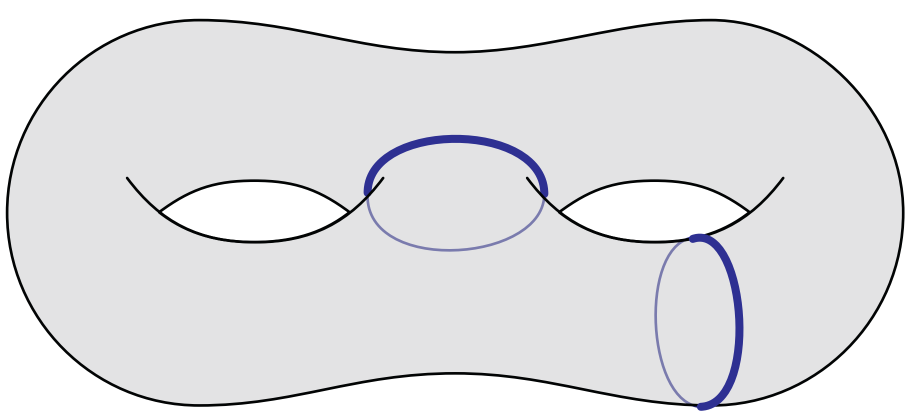
- Intuitively describes how a cycle wraps around surface features
- Two even subgraphs η and η' are ***homologous*** if $\eta \oplus \eta'$ is separating



- Homology partitions even subgraphs into 2^{2g} homology (equivalence) classes

More Homology

- An even subgraph is ***$\mathbb{Z}_2$-minimal*** if it is the smallest in its homology class
- Can find a $\mathbb{Z}_2$-minimal even subgraph in $g^{O(g)}\, n \log \log n$ time for any homology class [Italiano *et al.* '11]



The Ways to Separate

- Minimum separating subgraphs match one of two criteria:
  - Contractible simple cycle
  - Not a contractible simple cycle

The Ways to Separate

✓ ?

- We describe two algorithms
- Each returns a non-empty separating subgraph or fails
- One algorithm will find the minimum separating subgraph
- Return the smallest result

The Ways to Separate

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  - Contractible simple cycle
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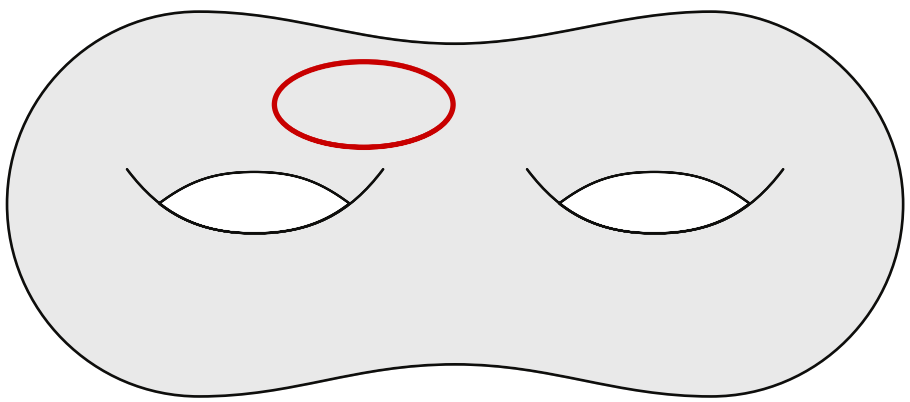
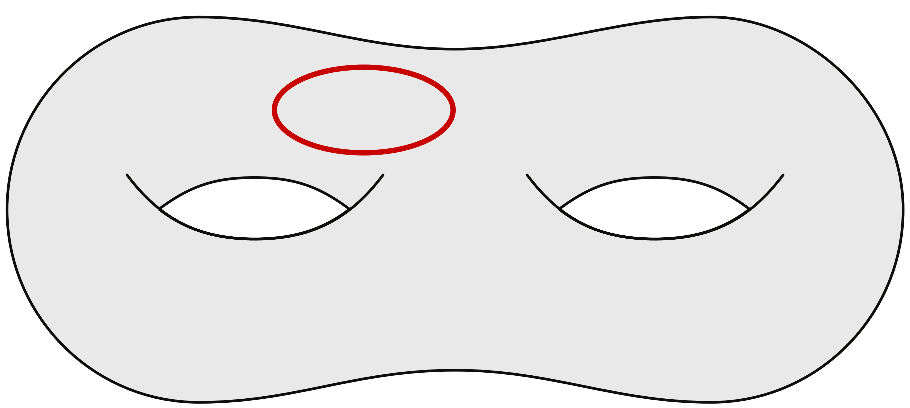


Contractible Cycle

- Can be continuously shrunk to a point
- Bounds a disk



Slicing

- Operation to "slice" surface along a path and lower genus
- Duplicates all vertices and edges on path

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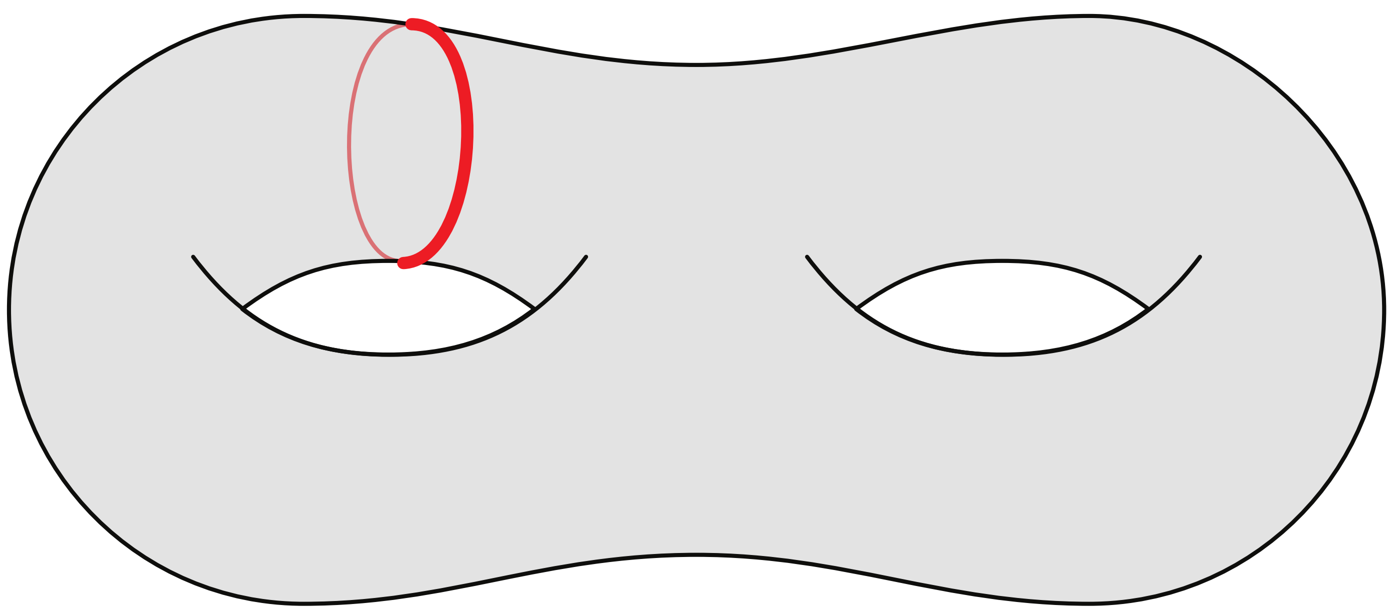

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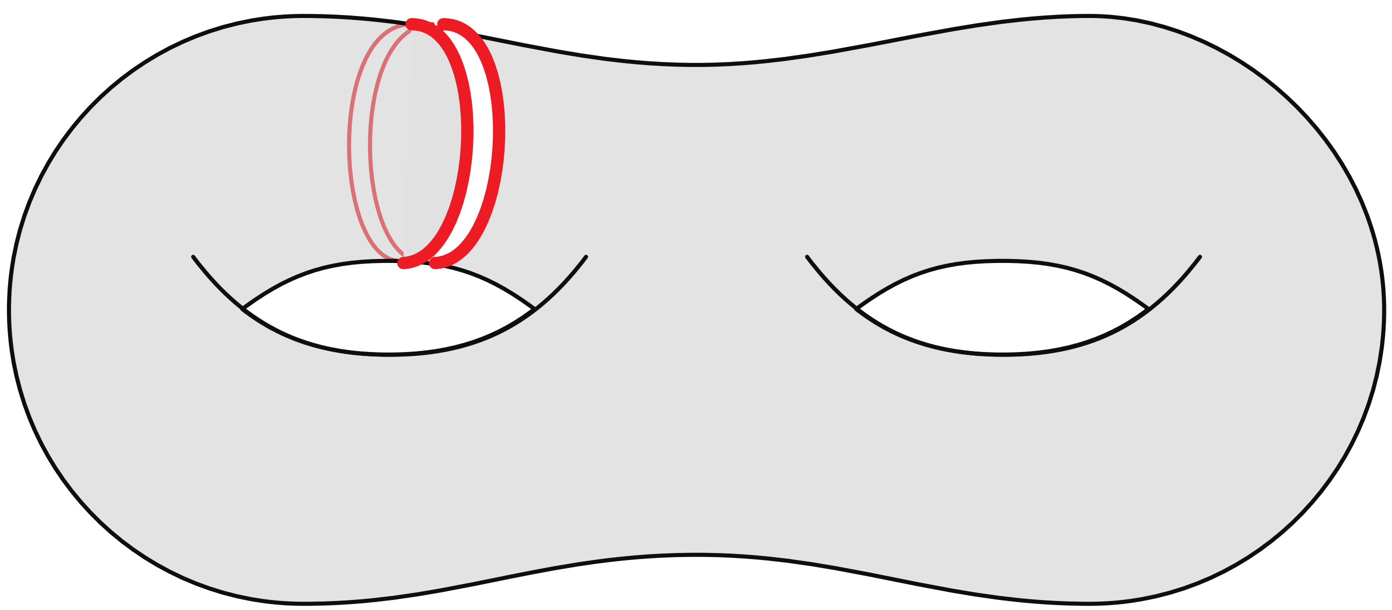
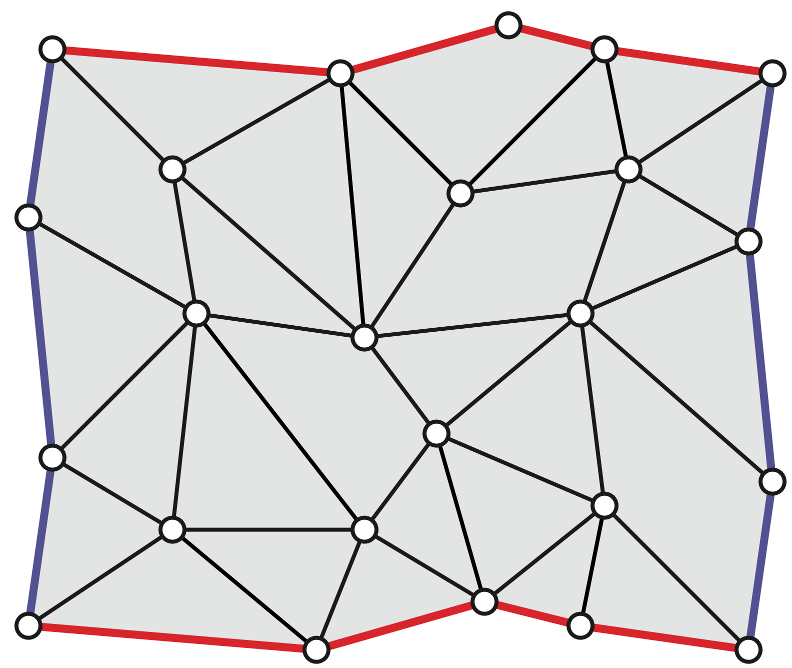


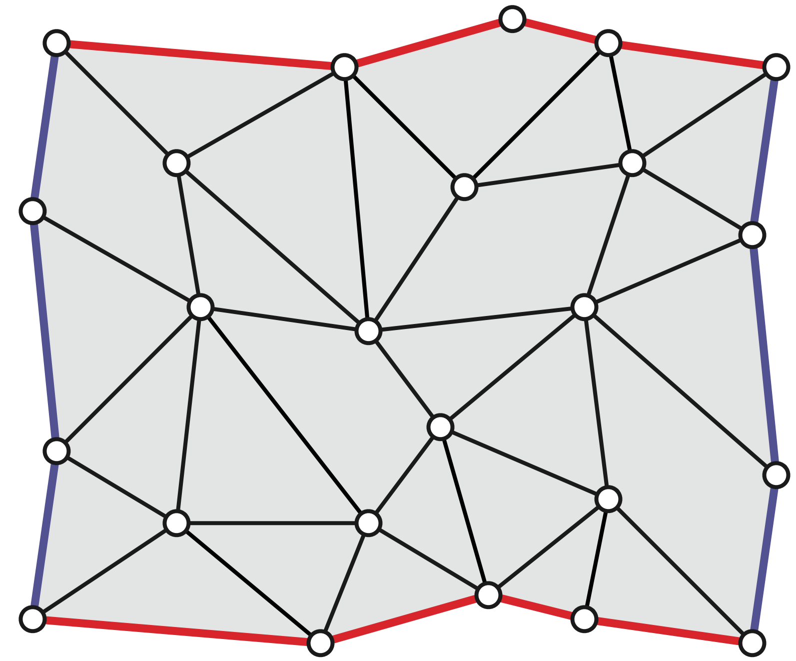
Slicing Away the Genus

- Slice along several paths to make graph planar without destroying minimum separating cycle [Cabello '10]
- Doable in $g^{O(g)}$ $n \log \log n$ time [Italiano *et al.* '11]



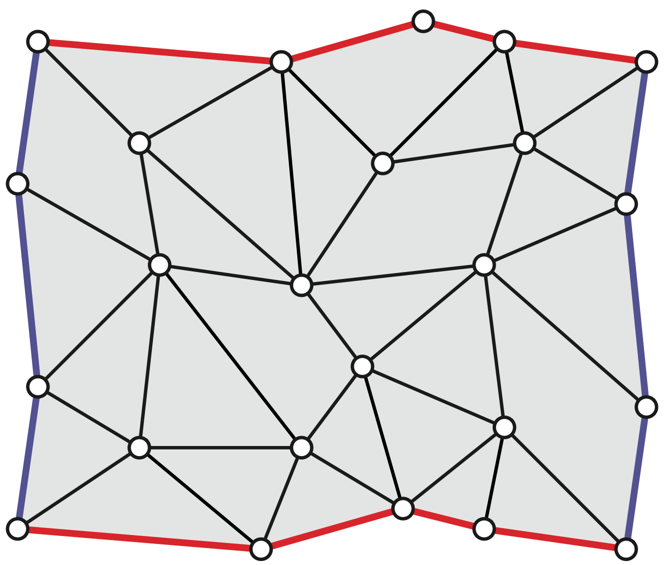
Searching the Plane

- Need to separate planar faces
- Could search directly for the shortest cycle that does not repeat vertices in original surface graph [Cabello '10]
- Takes quadratic time!

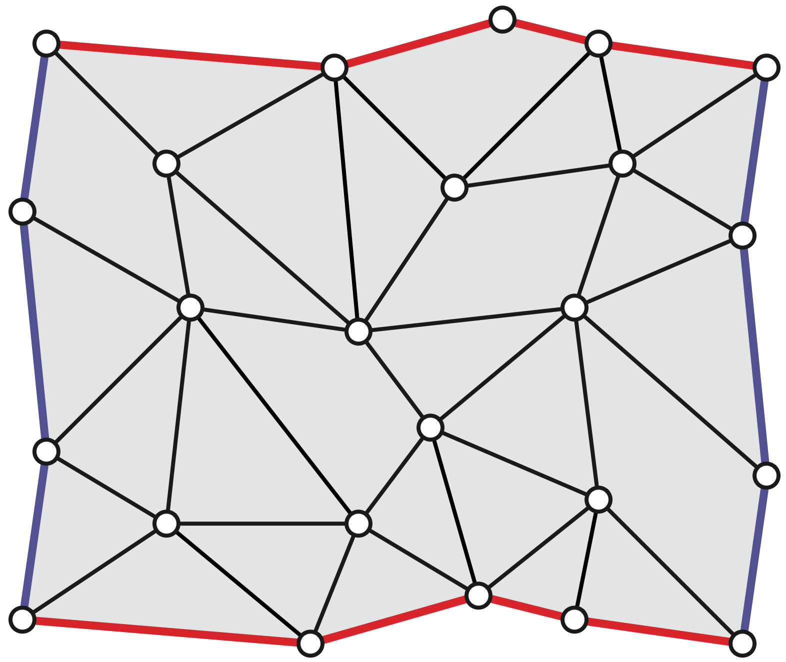


Close Enough

- Suffices to find minimum planar cycle separating *any* pair of faces
- Can find cycle in O(n log log n) time [Łącki and Sankowski '11]
- Cycle might enclose *all* the (non-boundary) faces

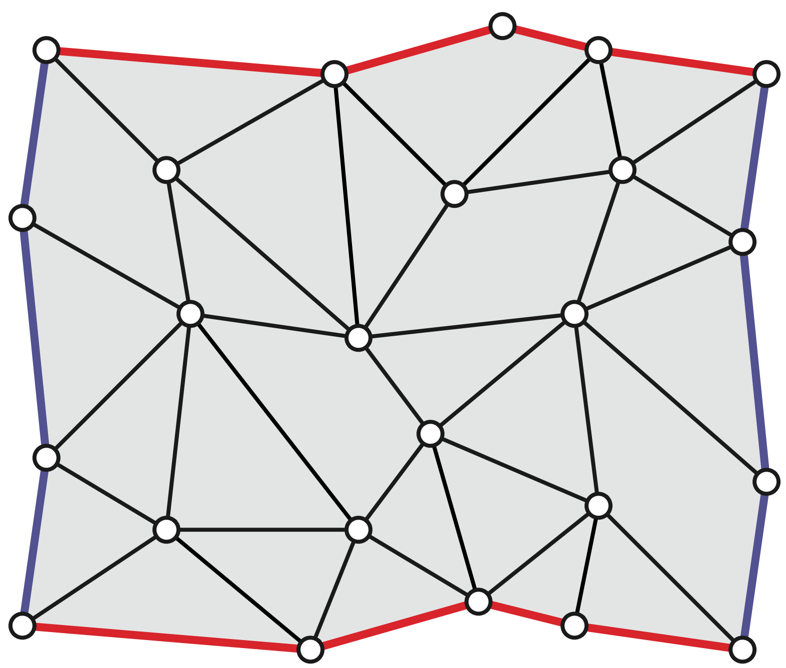


Forbidden Edge Pairs



Forbidden Edge Pairs

- Pick one sliced edge e with copies e_1 and e_2



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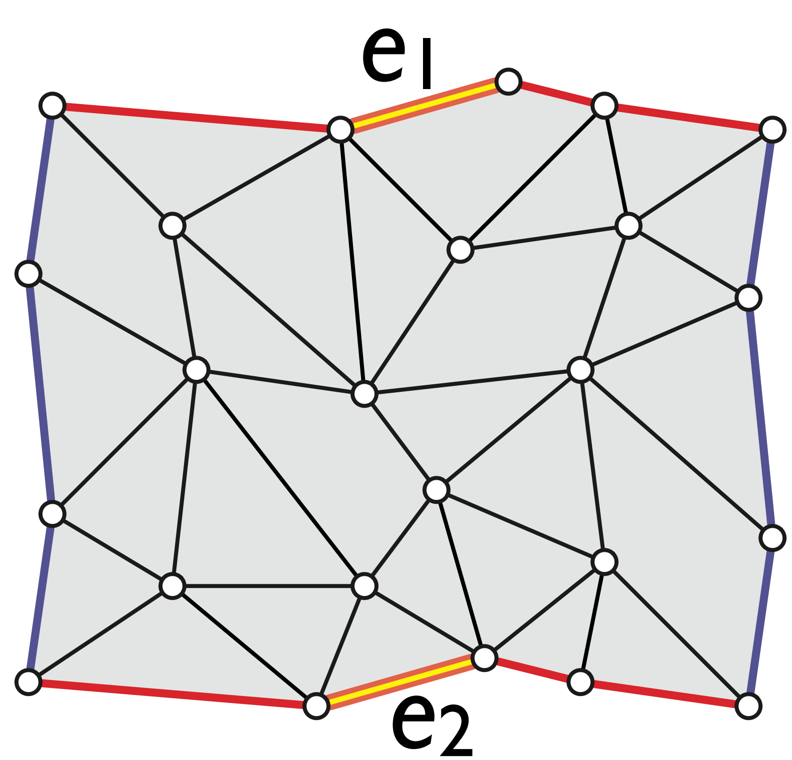

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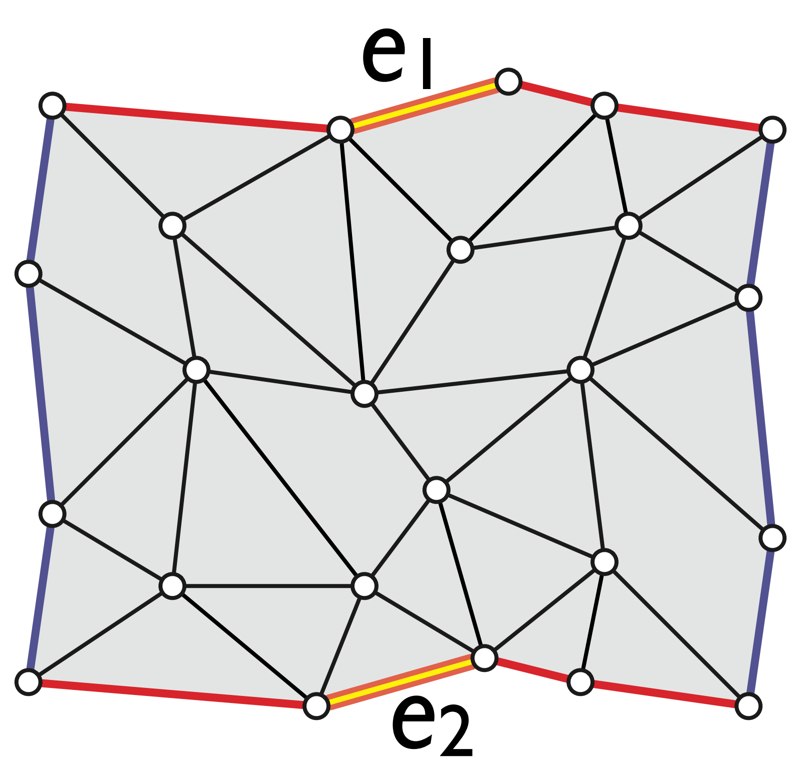
- Pick one sliced edge e with copies e_1 and e_2
- Find shortest cycle avoiding e_1 and the shortest cycle avoiding e_2



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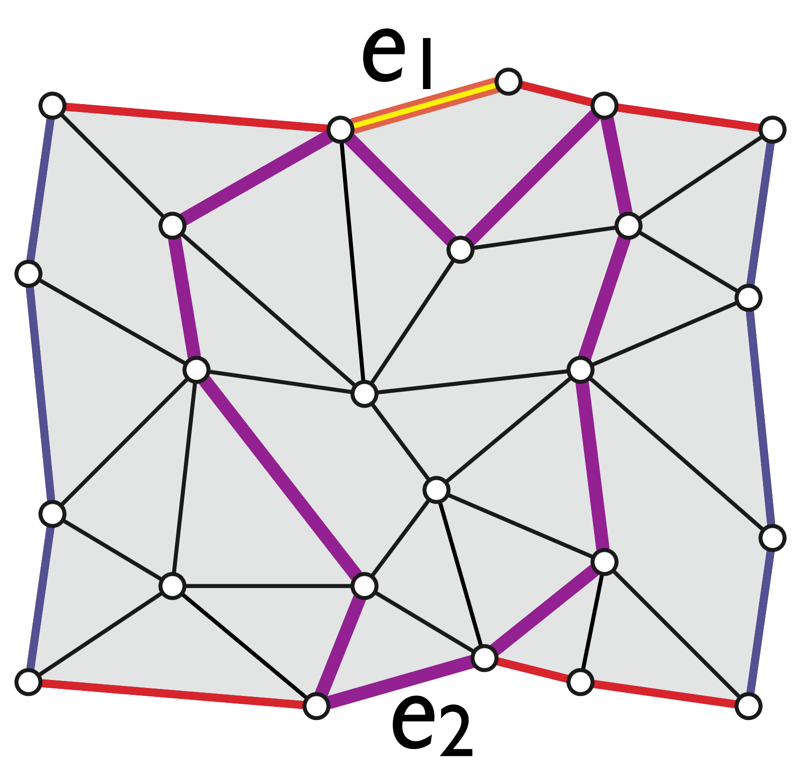

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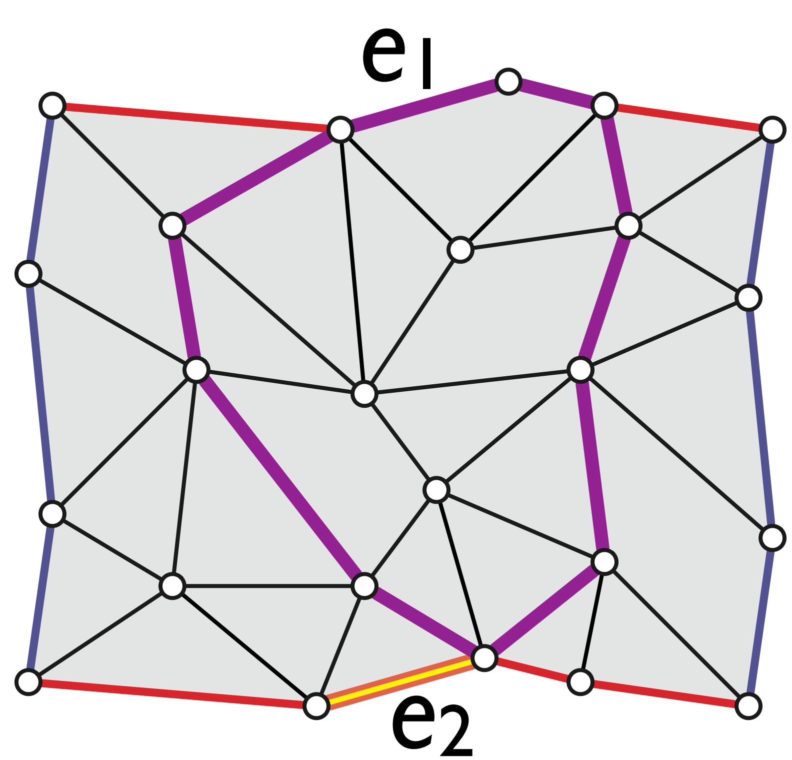
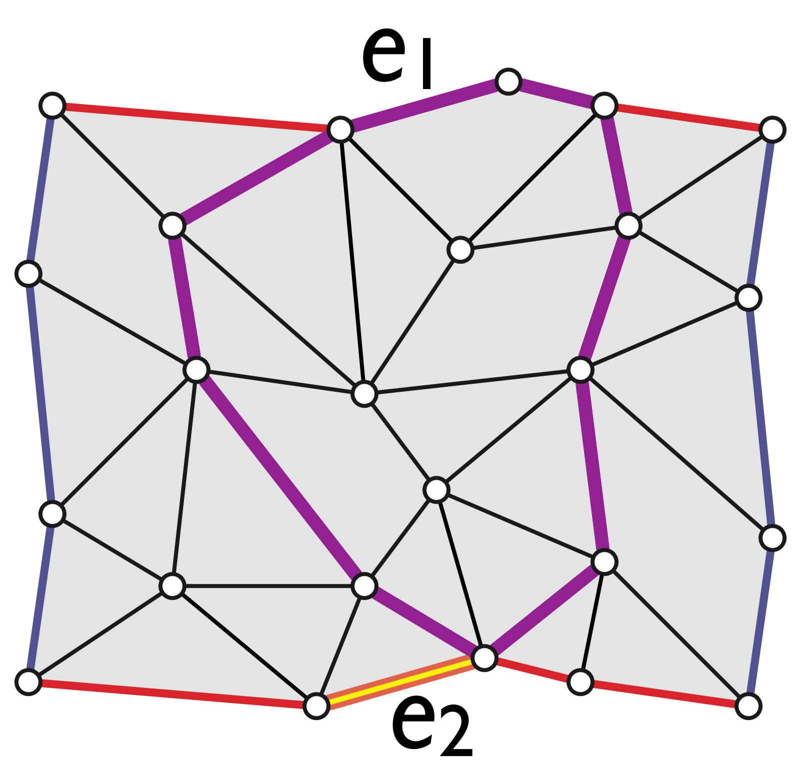


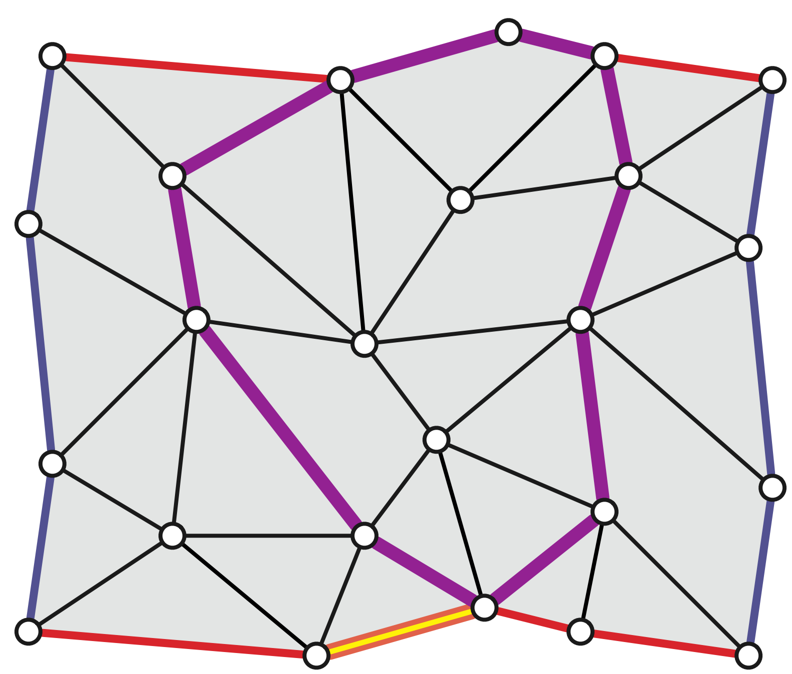
Forbidden Edge Pairs

- Pick one sliced edge e with copies e_1 and e_2
- Find shortest cycle avoiding e_1 and the shortest cycle avoiding e_2
- Return the smaller of the two results



Another Case Down

- Both cycles have at least one outside face so edges of cycle *will* separate some faces of surface graphs
- Shortest contractible cycle must avoid one copy of e anyway



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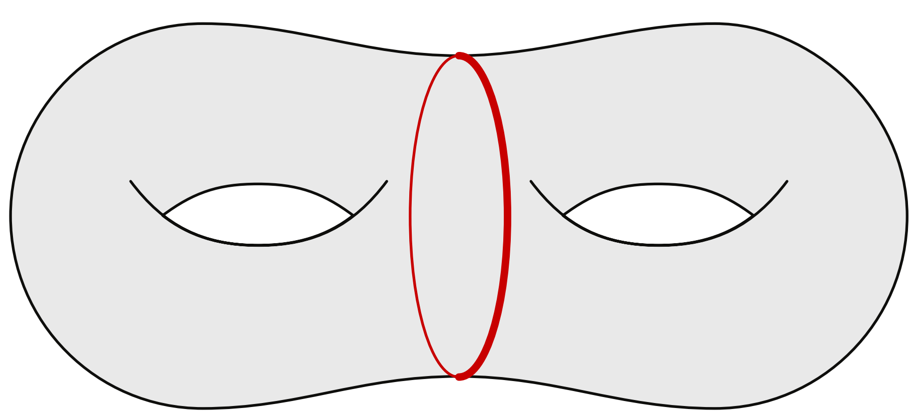
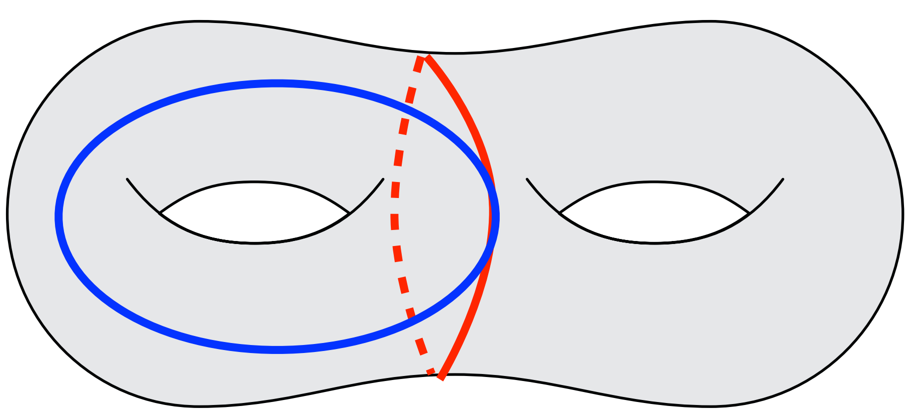
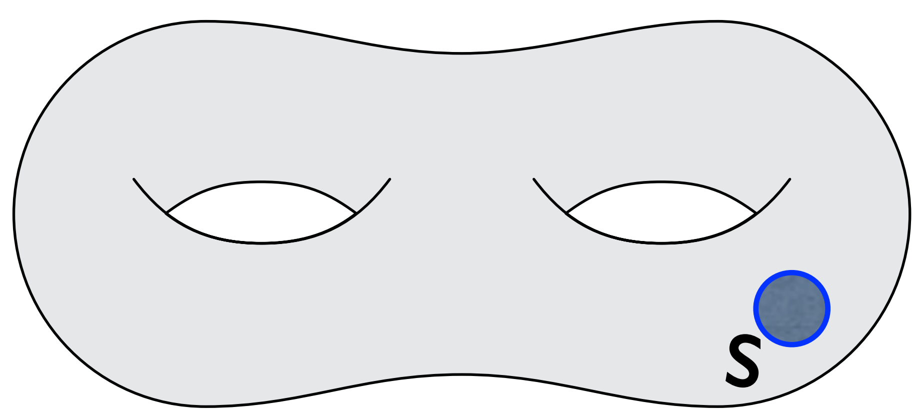


Stick to One Side

* There exist $\mathbb{Z}_2$-minimal even subgraphs on both sides of the minimum separating subgraph that do not cross the minimum separating subgraph

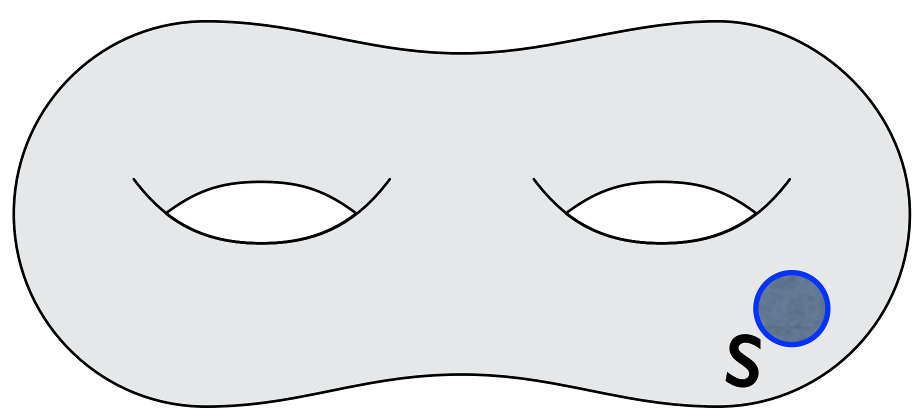


Finding a Pair of Faces



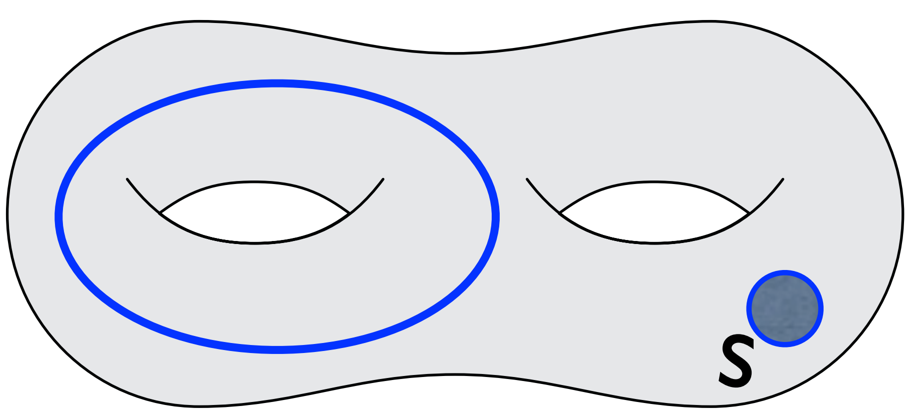
Finding a Pair of Faces

- Fix a face *s*

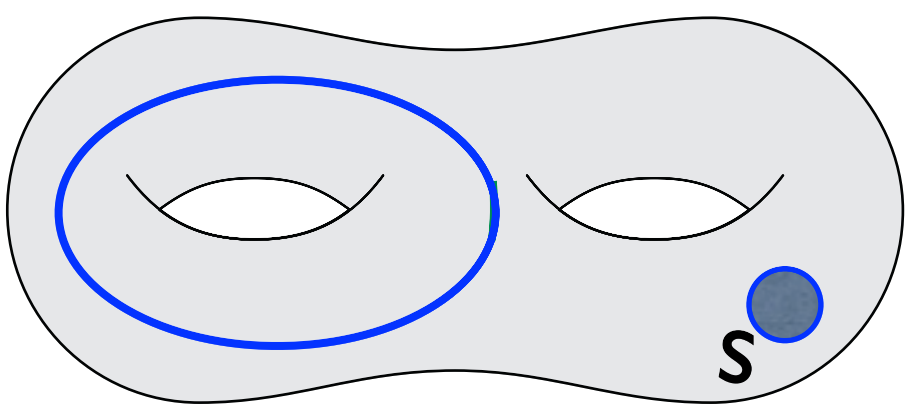


Finding a Pair of Faces

- Fix a face *s*
- Suppose we have some cycle that is separated from *s*

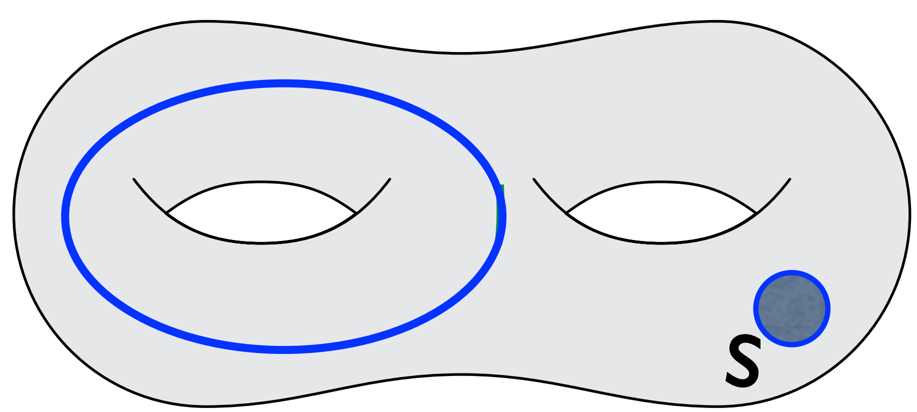


Two Cuts



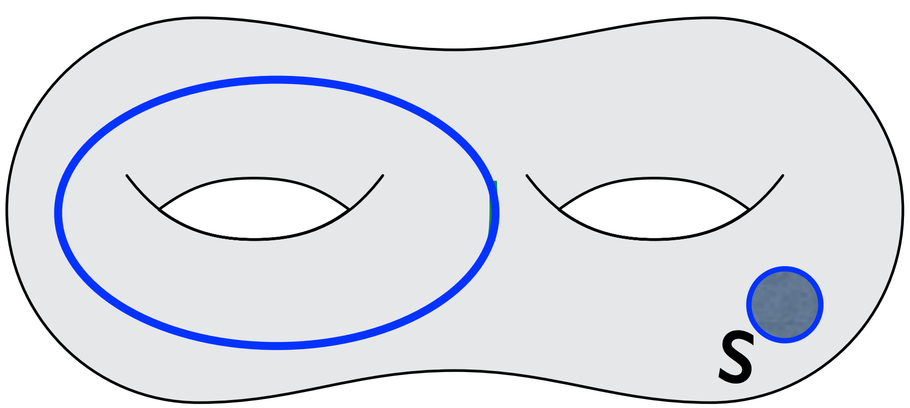
Two Cuts

- Pick an edge on the cycle with incident faces t_1 and t_2



Two Cuts

- Pick an edge on the cycle with incident faces t_1 and t_2
- At least one of t_1 and t_2 must be separated from s
- Return the smaller of s,t_1 and s,t_2 minimum cuts



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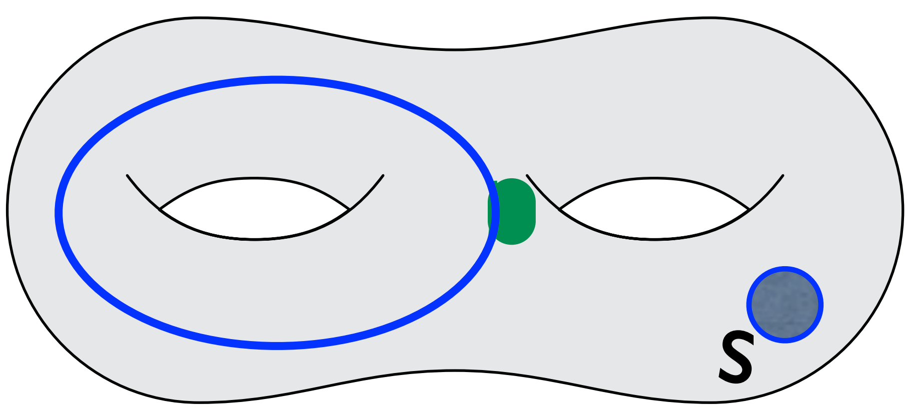

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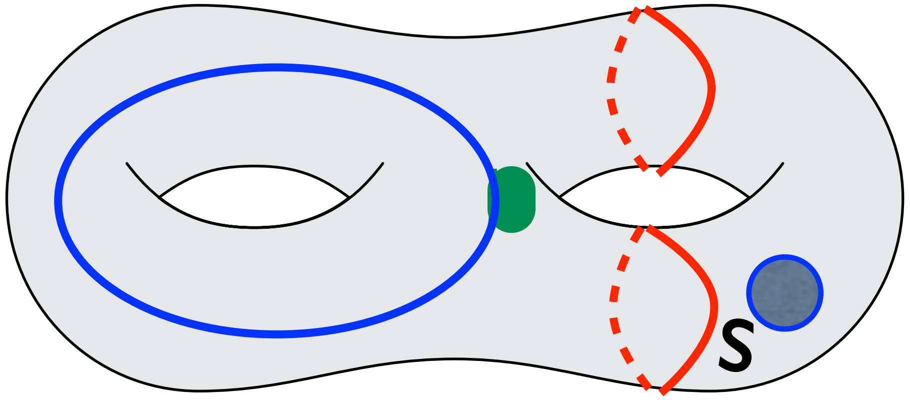

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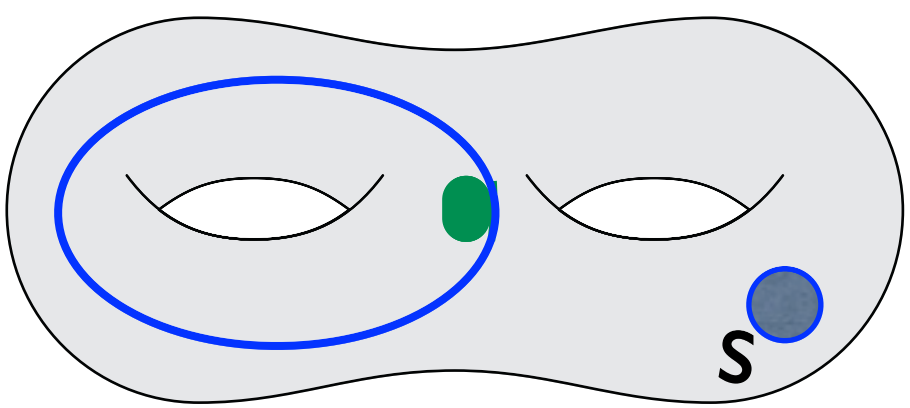

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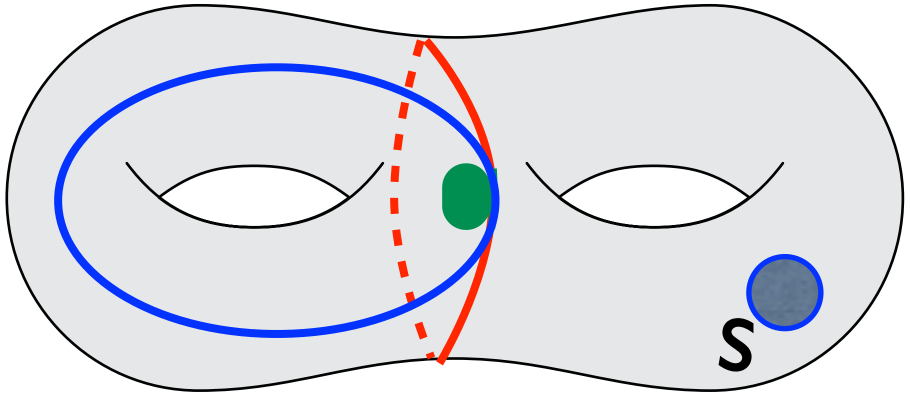

Final Case Closed

- Return the smallest separating subgraph found after taking cycles from all $2^{O(g)}$ $\mathbb{Z}_2$-minimal even subgraphs
- $g^{O(g)}$ $n \log \log n$ time spent in total

The Ways to Separate

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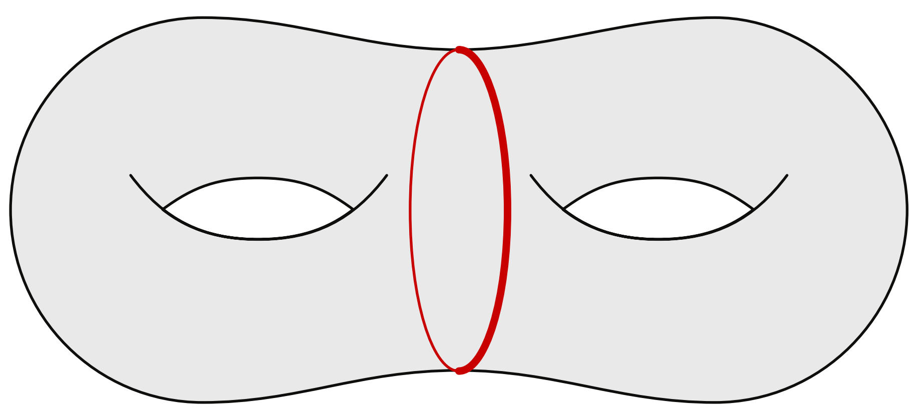

Conclusion

- Return the smallest result found of both cases
- We can find smallest separating subgraphs and minimum cuts in $g^{O(g)}\, n \log \log n$ time

Open Problems

- We conjecture a $O(g^k\, n \log \log n)$ time algorithm exists for some small constant k
  - NP-hard to find arbitrary $\mathbb{Z}_2$-minimal even subgraphs
- Is finding the smallest ***separating simple cycle*** with no repeating vertices FPT in g?
  - Problem is NP-hard, but reduction uses a surface with polynomial complexity

Thank you