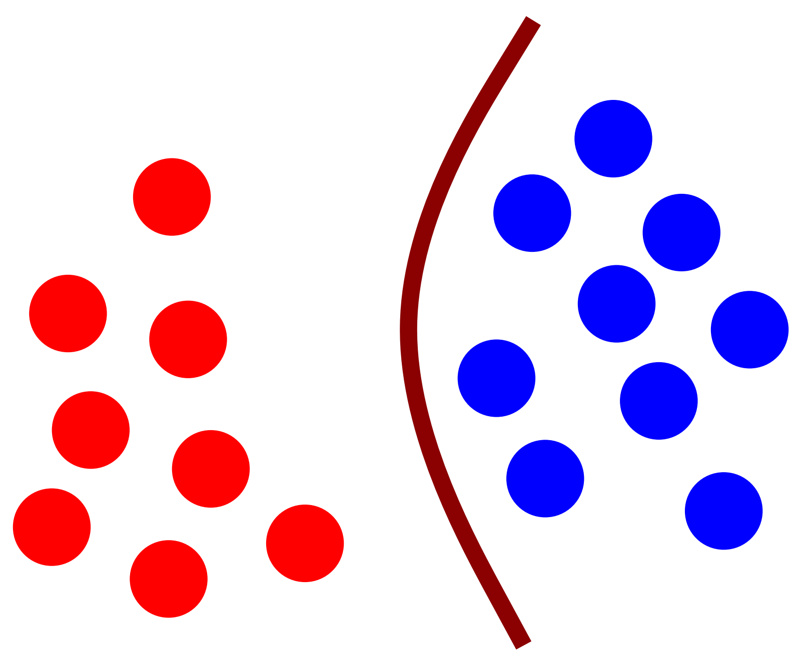
Global Minimum Cuts in Surface Embedded Graphs

Jeff Erickson, **Kyle Fox**, and Amir Nayyeri

Minimum Cut

- Partition vertices of undirected graph into two non-empty sets
- Minimize weight of edges between sets



Minimum Cuts

- $O(nm + n^2 \log n)$ deterministic [Nagamochi, Ibaraki '92]
- $O(m \log^3 n)$ randomized [Karger '00]

Planar Cuts

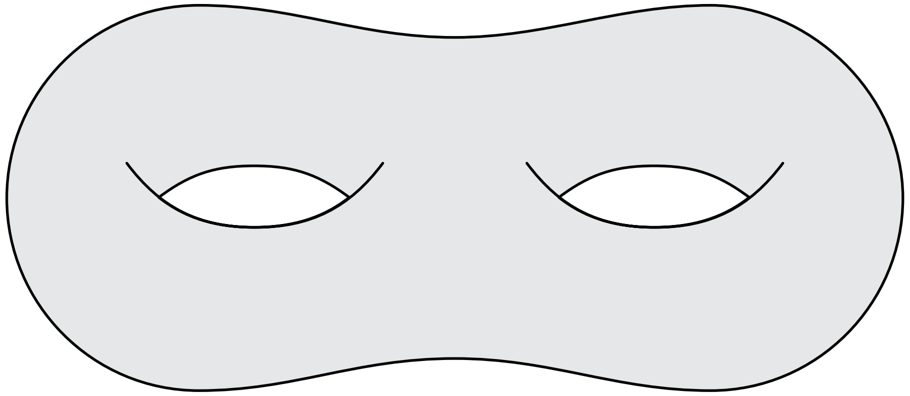
- $O(n^2 \log n)$ deterministic [N, I '92]
- $O(n \log^3 n)$ randomized [K '00]
- $O(n \log^2 n)$ deterministic [Chalermsook, Fakcharoenphol, Nanongkai '04]
- $O(n \log n \log \log n)$ deterministic [Italiano *et al.* '11]
- $O(n \log \log n)$ deterministic [Łącki and Sankowski '11]

Planar Graphs

- Supports thesis of "planar = fast"
- Applies to *s*,*t*-cuts, flows, SSSPs, MSTs, *etc*.
- Numerous generalizations

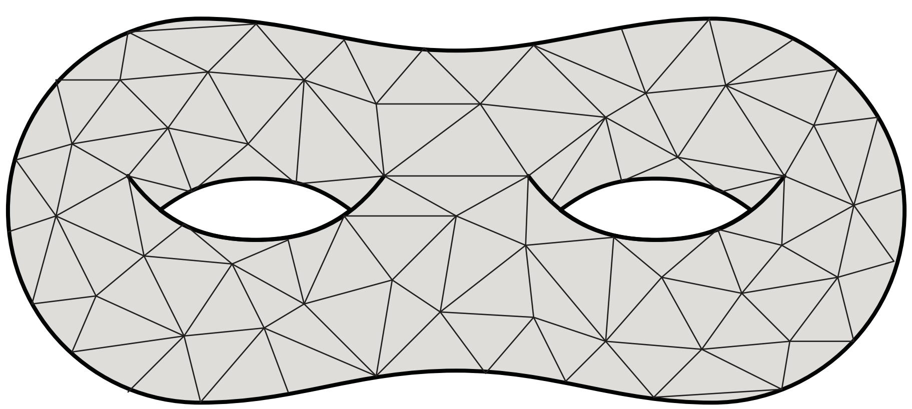
Surfaces

- 2-manifolds (with boundary)
- ***genus g***: max # of disjoint simple cycles whose compliment is connected
  = number of holes
  = number of handles attached to sphere



Surface Graphs

- Generalizes planar graphs
- *Most* planar results generalize easily
- *s*,*t*-cuts and flows only recently [Chambers, Erickson, Nayyeri STOC/SOCG ’09; Italiano et al. ’11]



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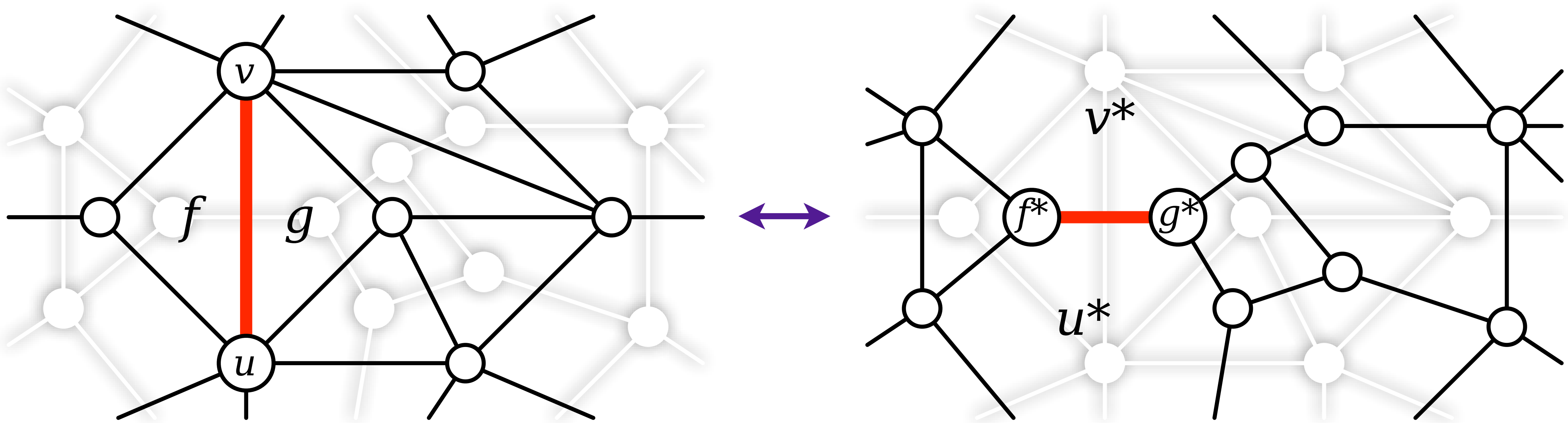
- Can compute minimum cuts in surface graphs in $g^{O(g)}\, n \log\log n$ time
- Matches time bound of Łącki and Sankowski
- Requires an orientable surface

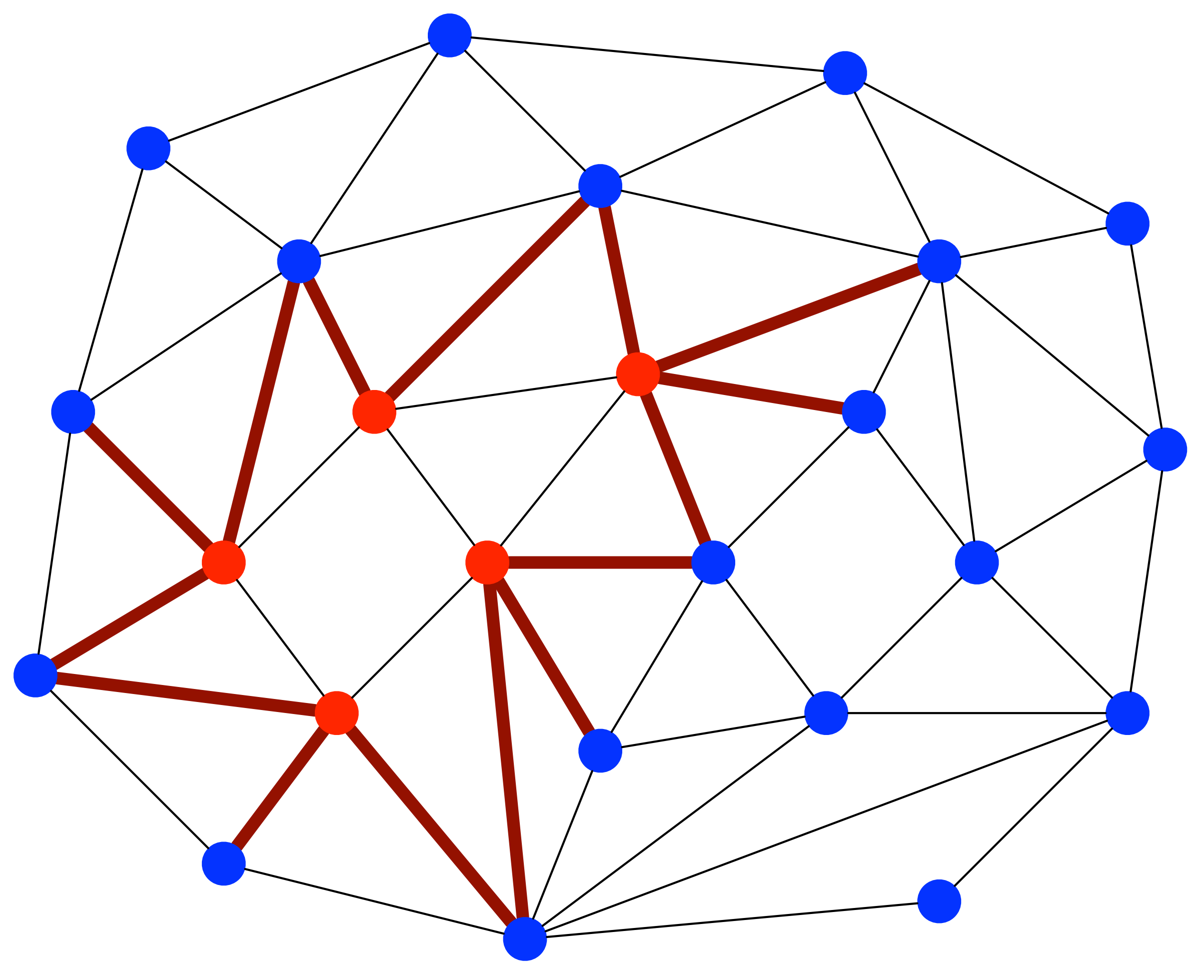
Back to the Plane

- Recall the dual…



Back to the Plane

- Minimum cut is dual to a cycle



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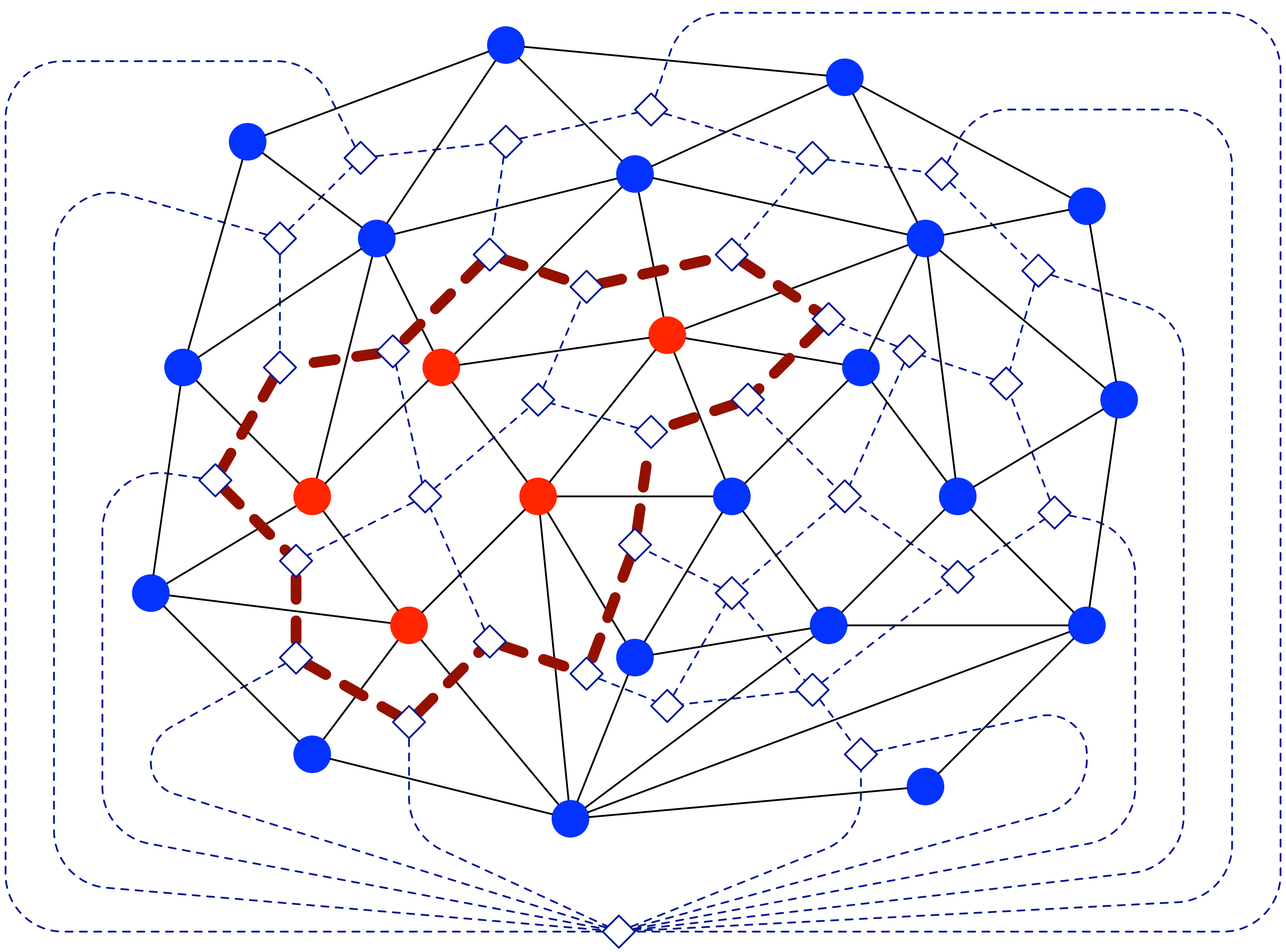

Chalermsook, *et al.*

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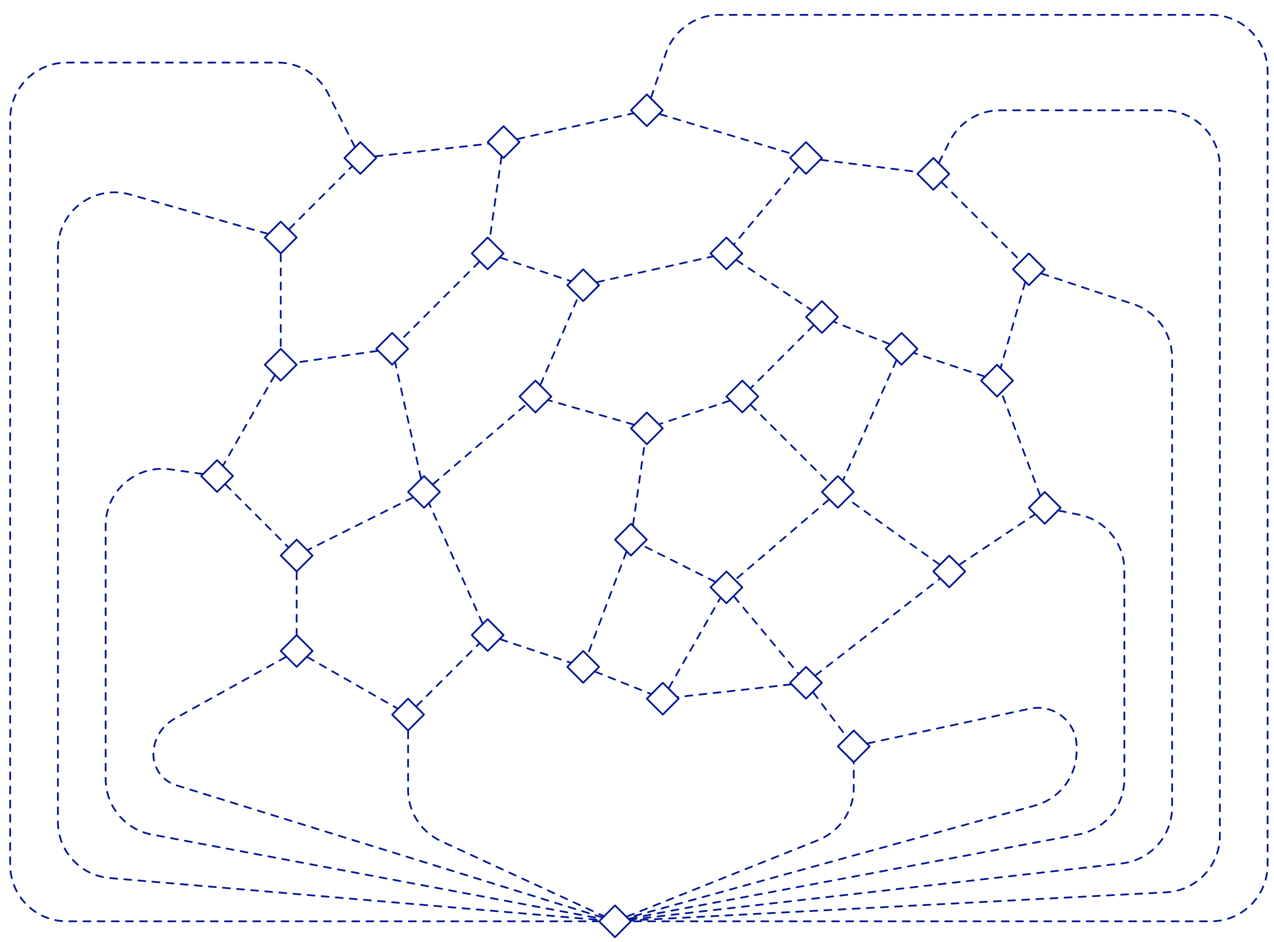

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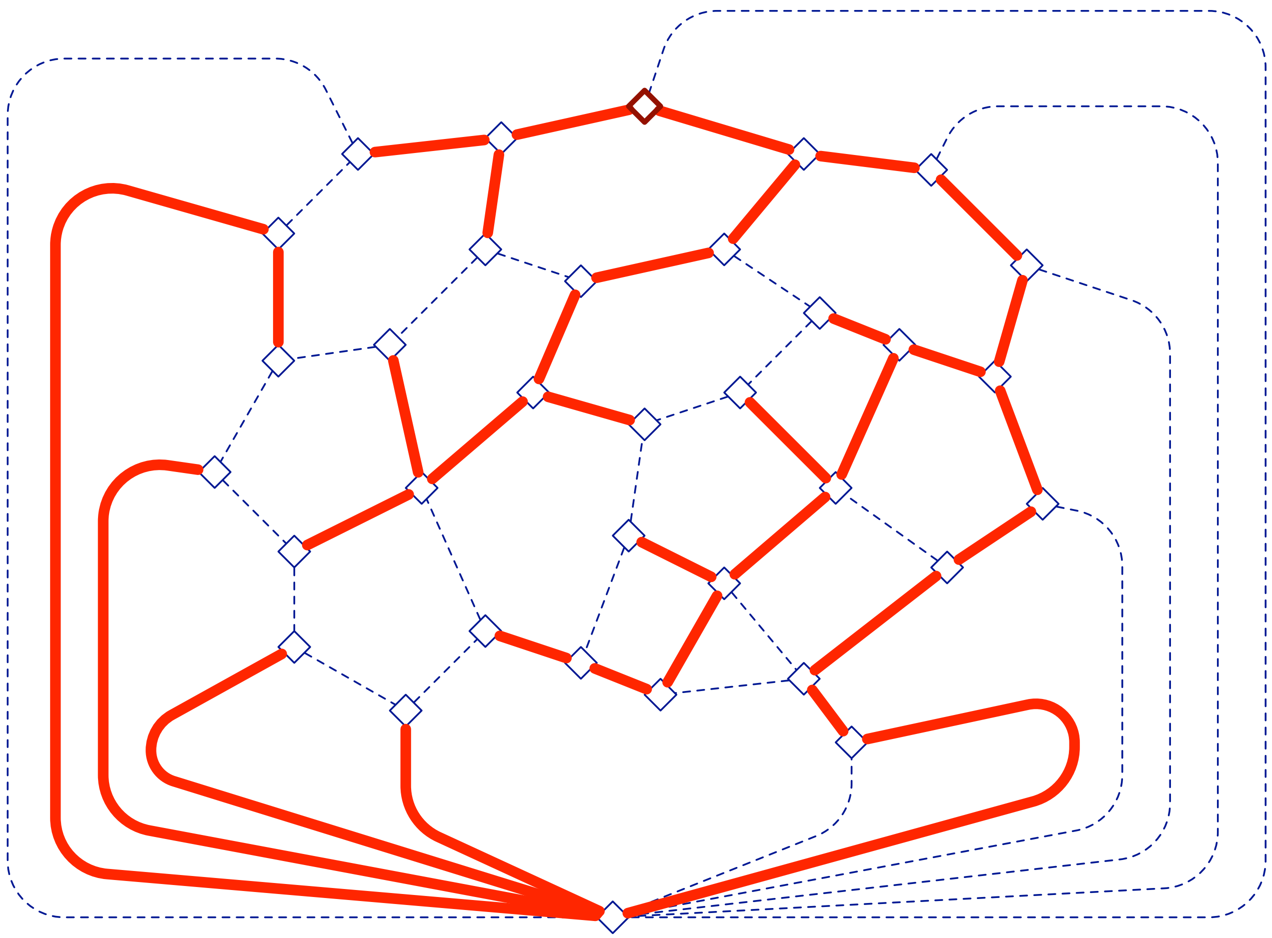

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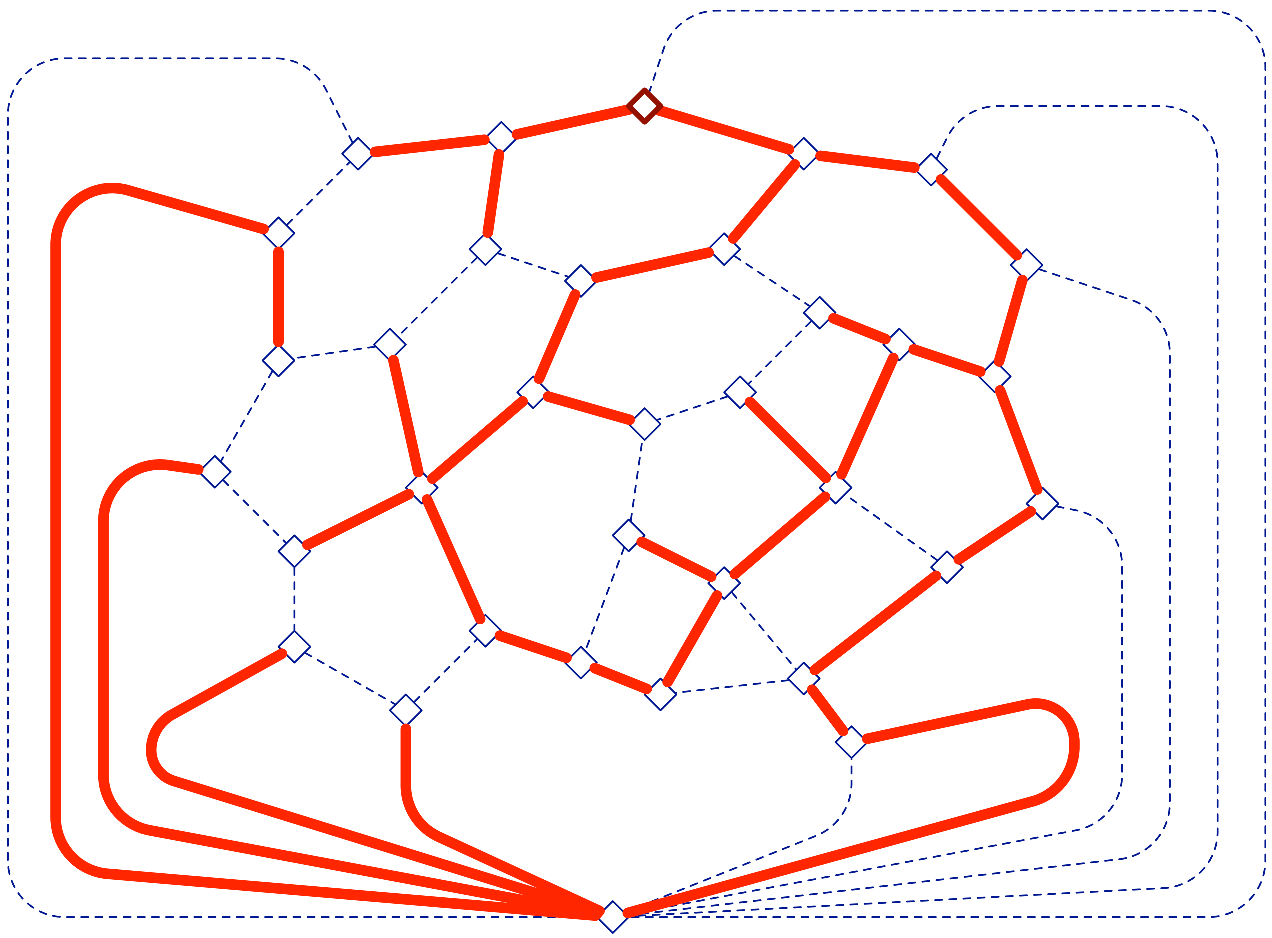

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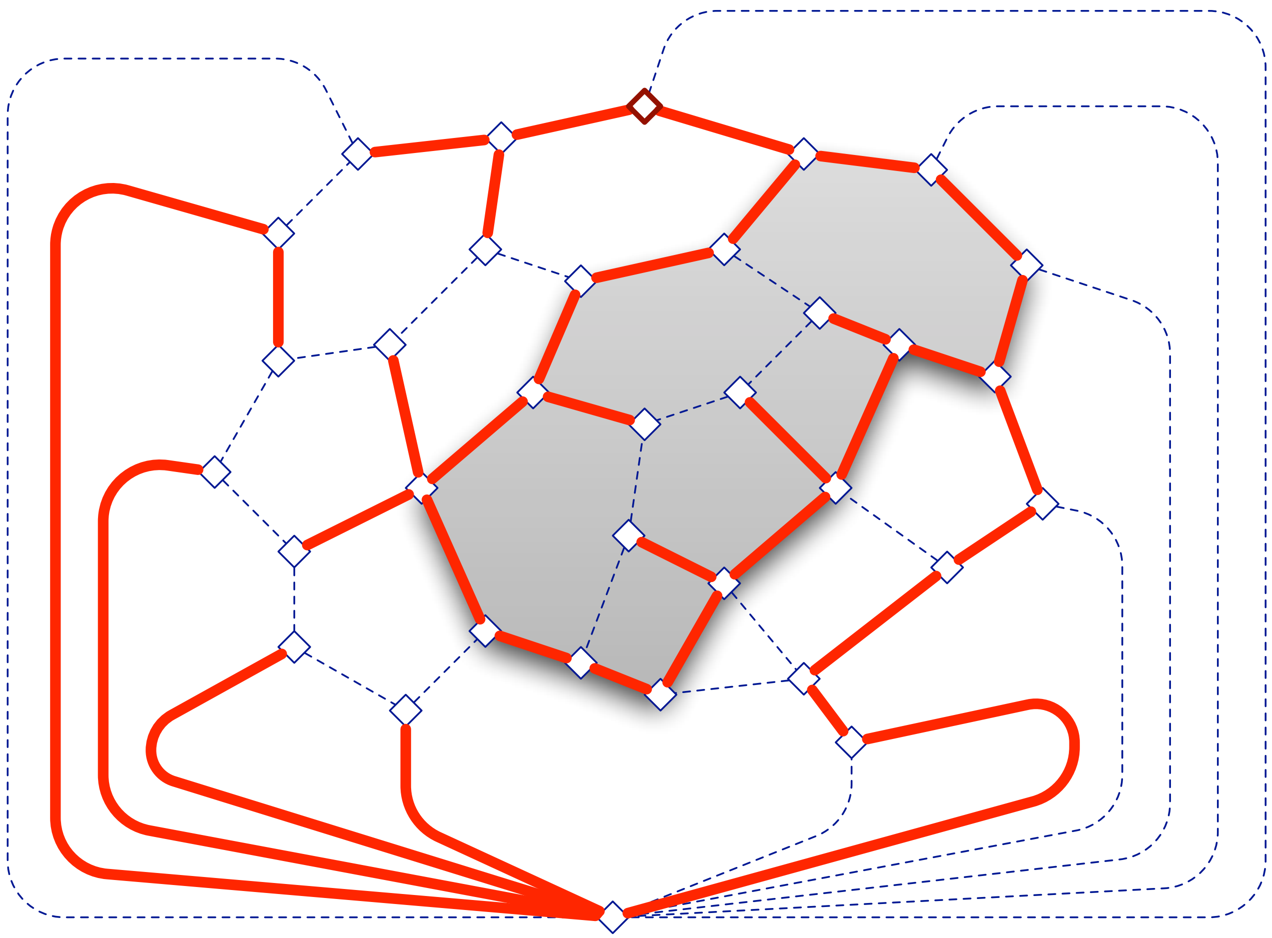
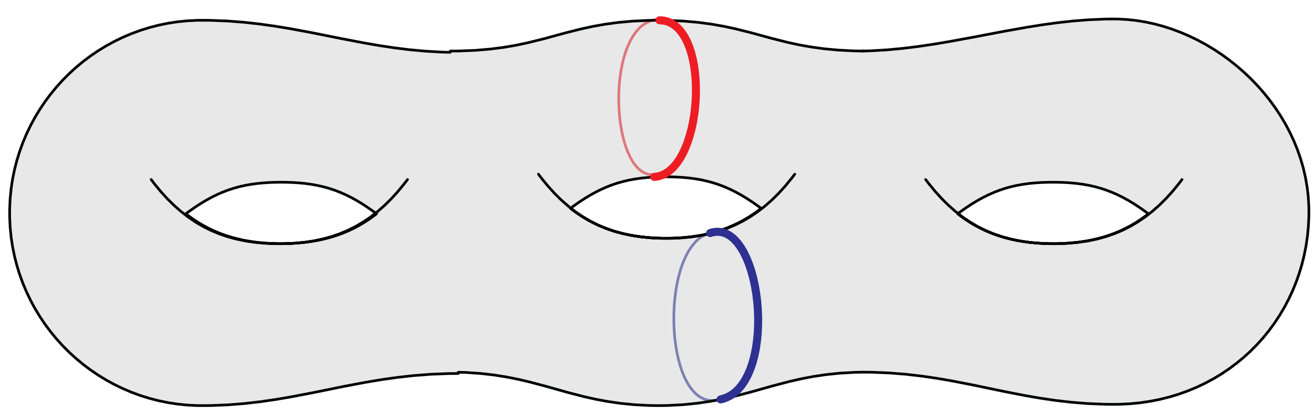


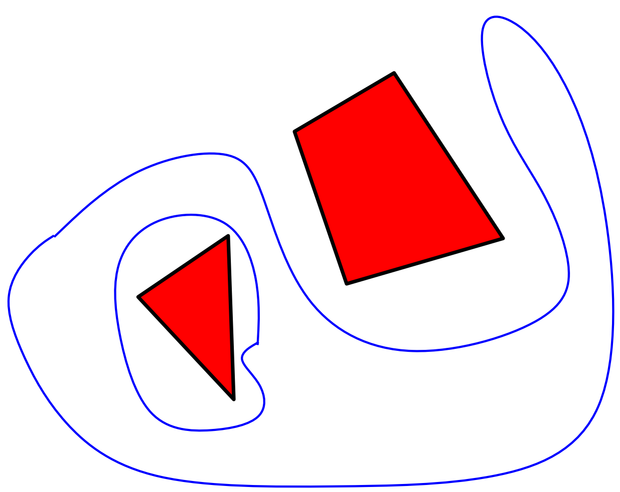
How to Generalize?

- Cycles maybe not separate dual faces
- Minimum cut may have multiple components

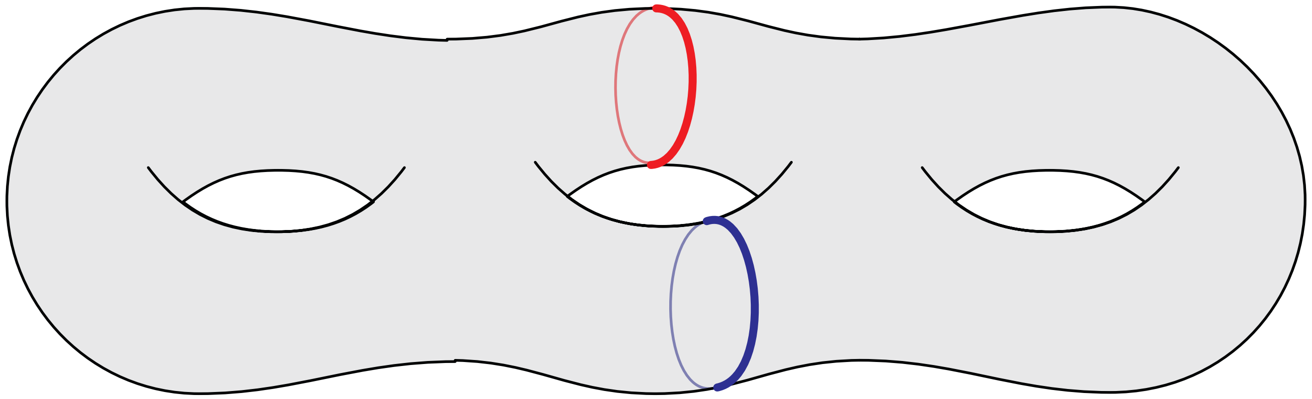


Homotopy

- Two curves are ***homotopic*** if one can be continuously deformed to become the other
- A cycle is ***contractible*** if it is homotopic to a point

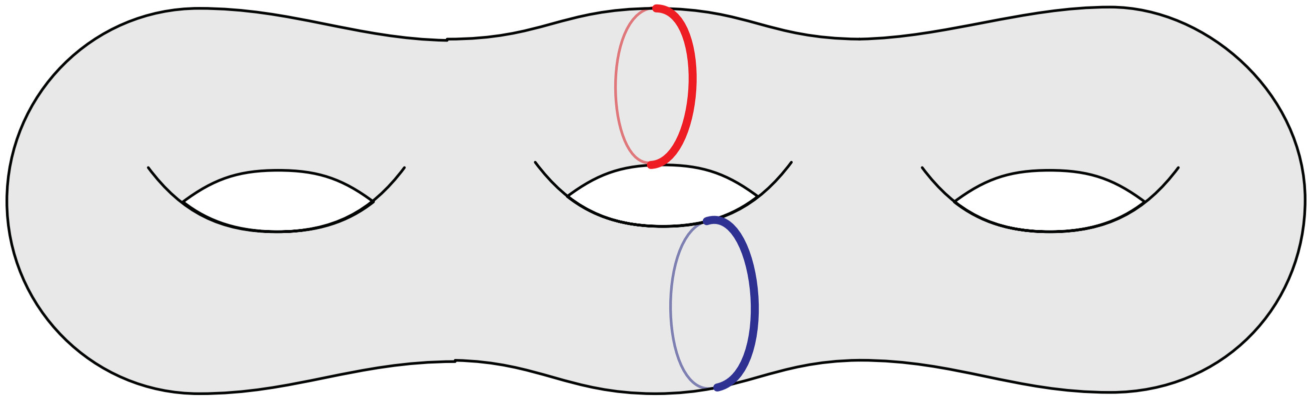


Homology



- An ***even subgraph*** has even degree on each node
- An even subgraph is ***separating*** if it removing it from the surface disconnects the surface
- Separating subgraphs form the boundary of a subset of faces

Homology

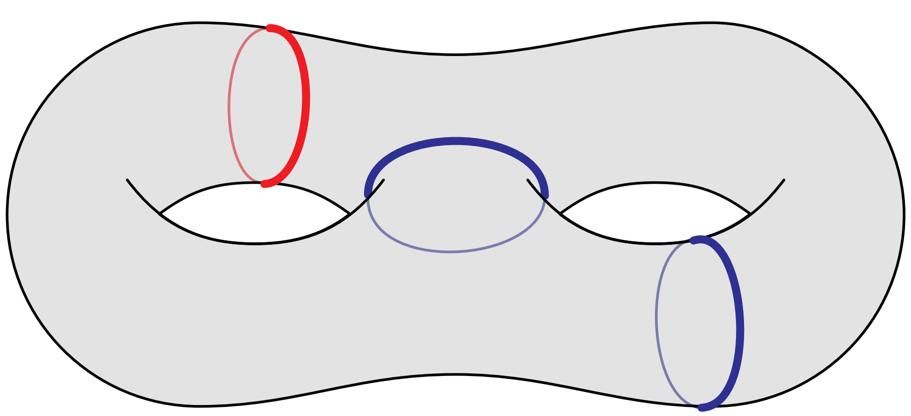


✴ The minimum cut is *dual* to a separating subgraph with one or more faces on each side of the separation

- Analogous to main lemma of [Chambers *et al.* '09]

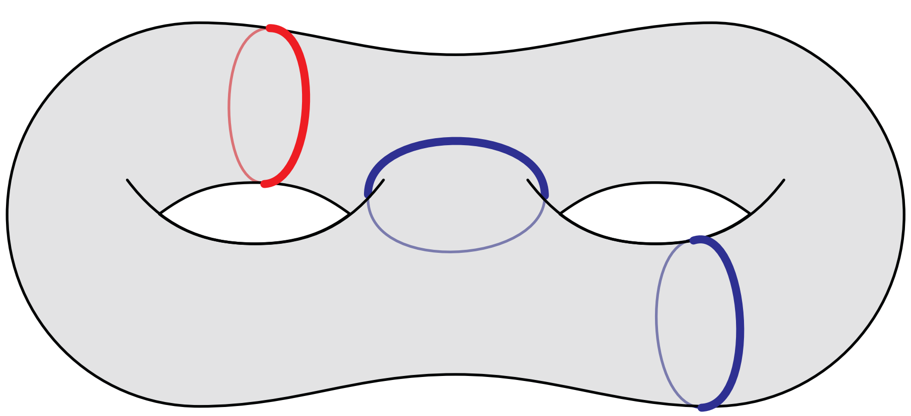
More Homology

- Two even subgraphs η and η' are ***homologous*** if η⊕η' is separating
- Homology partitions even subgraphs into 2^{2g} homology classes
- Intuitively counts number of times a cycles wraps around a feature (mod 2)



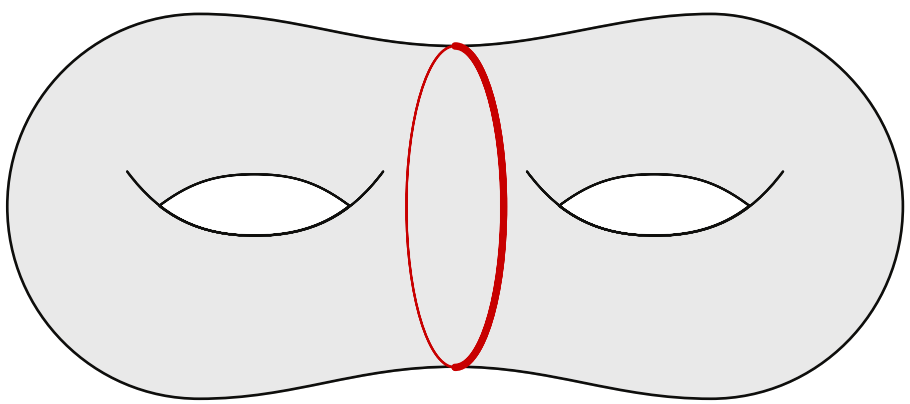
More Homology

- An even subgraph is $\mathbb{Z}_2$ ***-minimal*** if it is smallest in its homology class
- Can find a $\mathbb{Z}_2$-minimal even subgraph in $g^{O(g)}\, n \log \log n$ time for any *non-separating* homology class [Italiano *et al.* '11]



Final Definition

- A separating subgraph is ***splitting*** if it separates the surface into two components with genus



The Ways to Separate

- Min sep. subgraphs match *at least* one of three criterion:
  - Union of two nonempty edge-disjoint even subgraphs
  - Contractible simple cycle
  - Splitting subgraph

The Ways to Separate

- Describe three algorithms
- If min. sep. subgraph matches a certain criteria, then matching algorithm will find it

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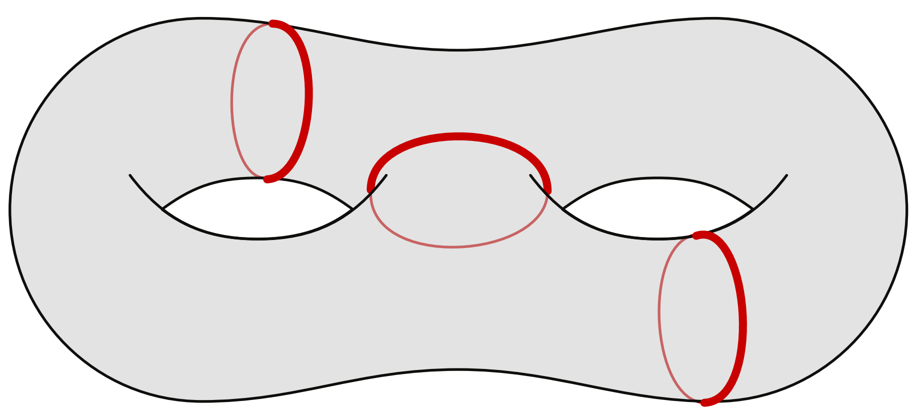

Two Even Subgraphs

$\mathbb{Z}_2$

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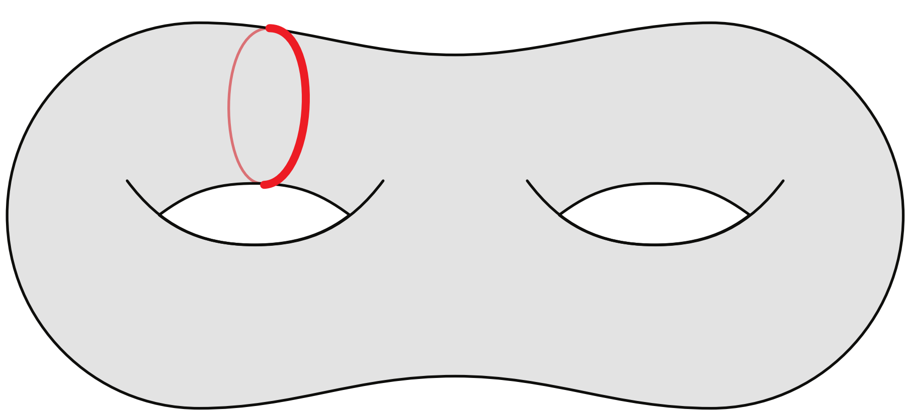

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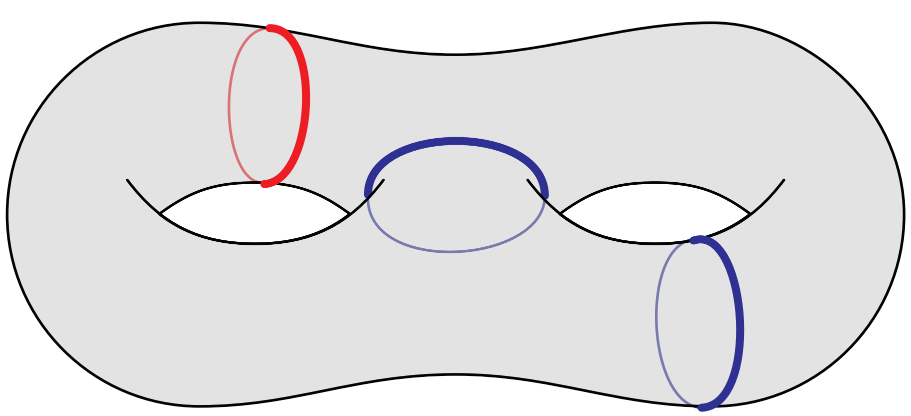

Two Even Subgraphs

- $2^{2g} - 1$ classes and $g^{O(g)}\ n \log \log n$ time per class
- $g^{O(g)}\ n \log \log n$ time total

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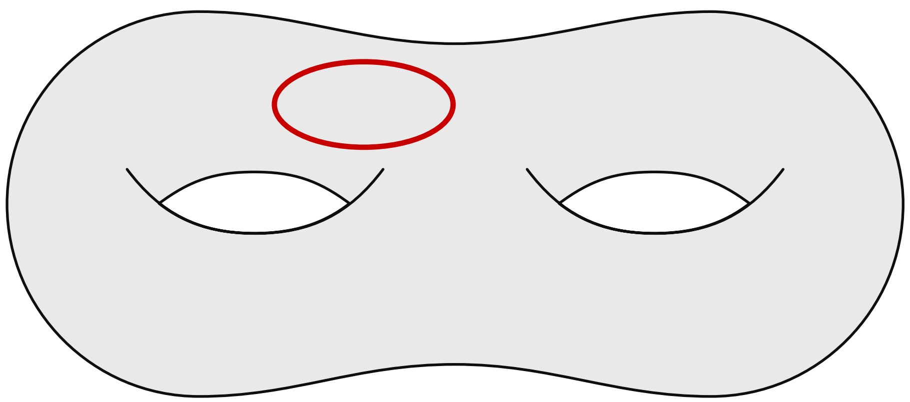
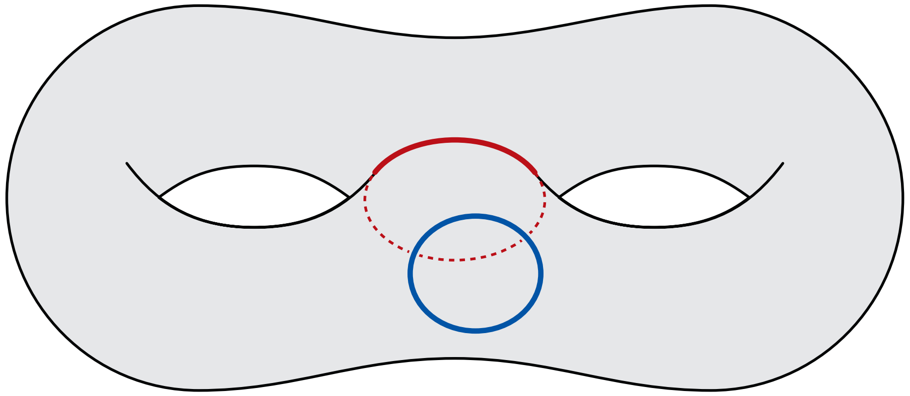


Contractible Cycle

- A walk is ***tight*** if it is shortest in its homotopy class
- The shortest contractible simple cycle does not cross any tight walks [Cabello '10]



Slicing Walks

- Operation to "slice" surface along a walk
- Duplicates all vertices and edges on walk.
- Incident edges to left and right stay on separate sides of the slice
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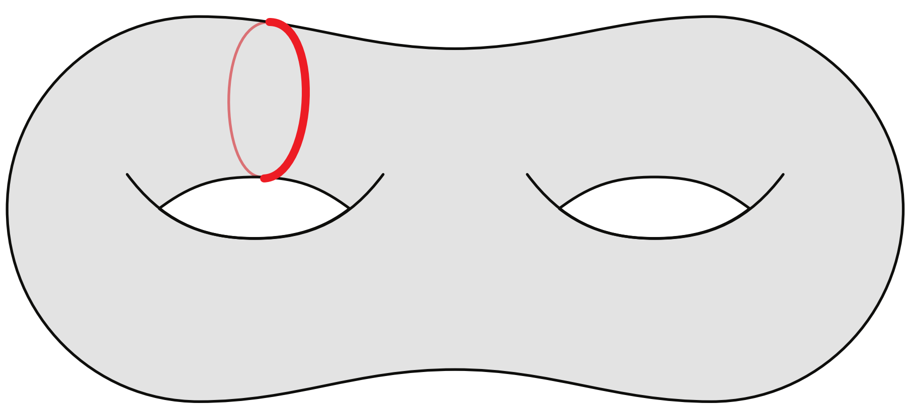

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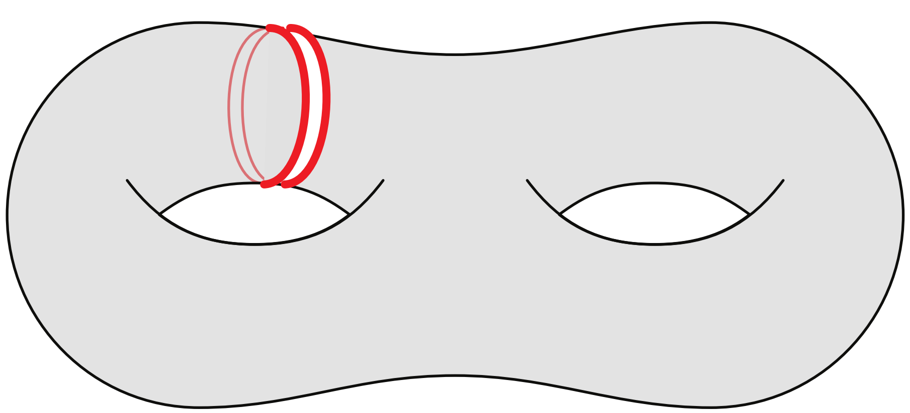
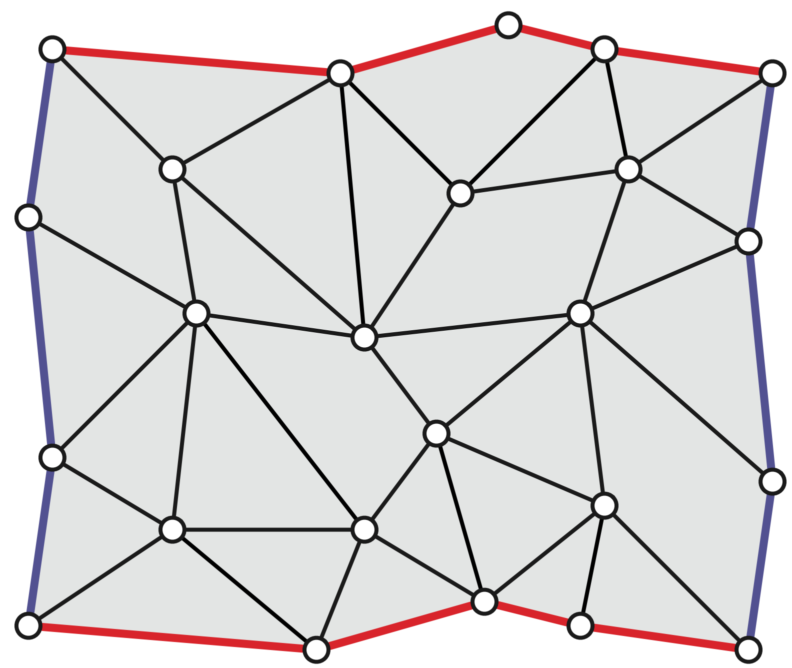


Slicing Away the Genus

- Slice along minimum separating cycle and several other walks to make graph planar
- Doable in $g^{O(g)}\, n \log \log n$ time [Italiano *et al.* '11]

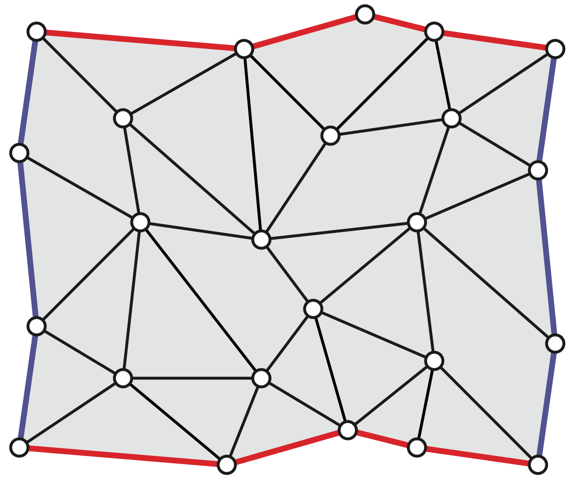


Searching the Plane

- Need to separate planar faces
- Could search directly for cycle that does not repeat vertices in original surface graph [Cabello '10]
- Takes quadratic time!

Close Enough

- Suffices to find minimum planar cycle separating *any* pair of faces - even if original vertices repeat
- Can find cycle in $O(n \log \log n)$ time [Łącki and Sankowski '11]
- Cycle might exclude only the boundary



Forbidden Edge Pairs

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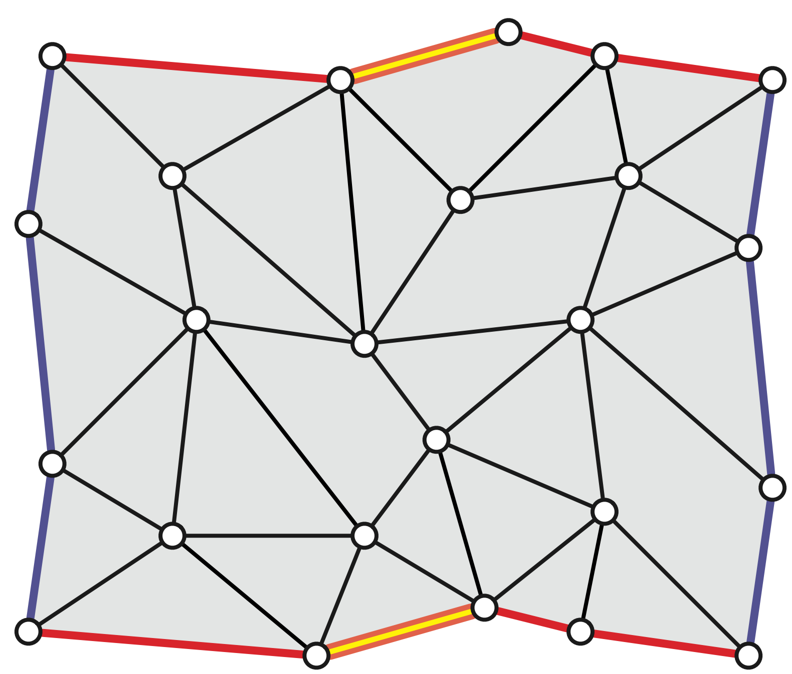
- Pick a sliced edge e with copies e_1 and e_2



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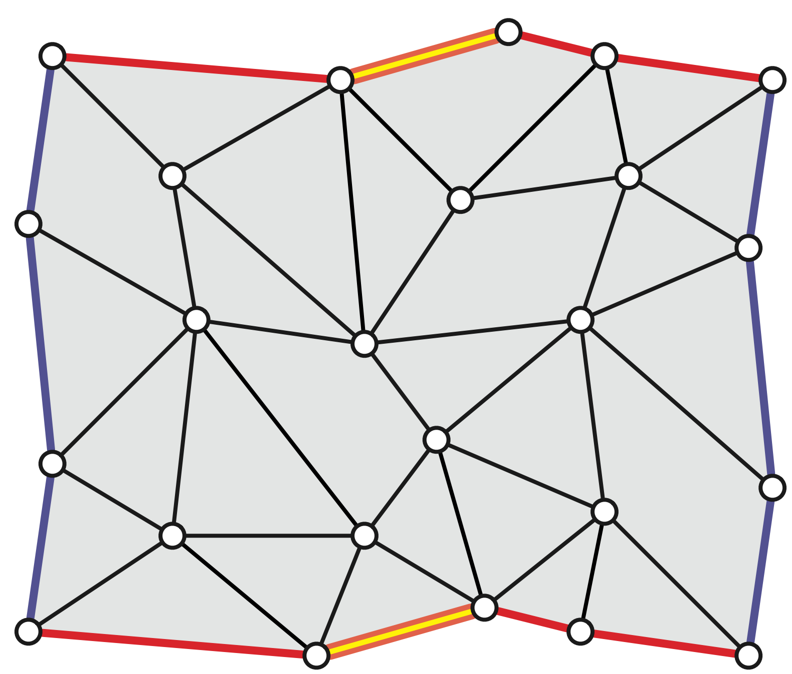

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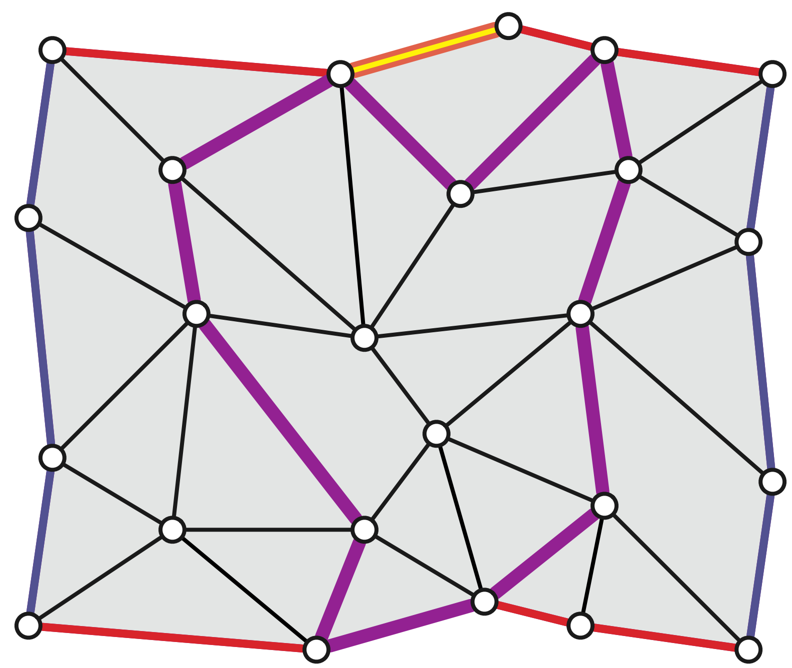

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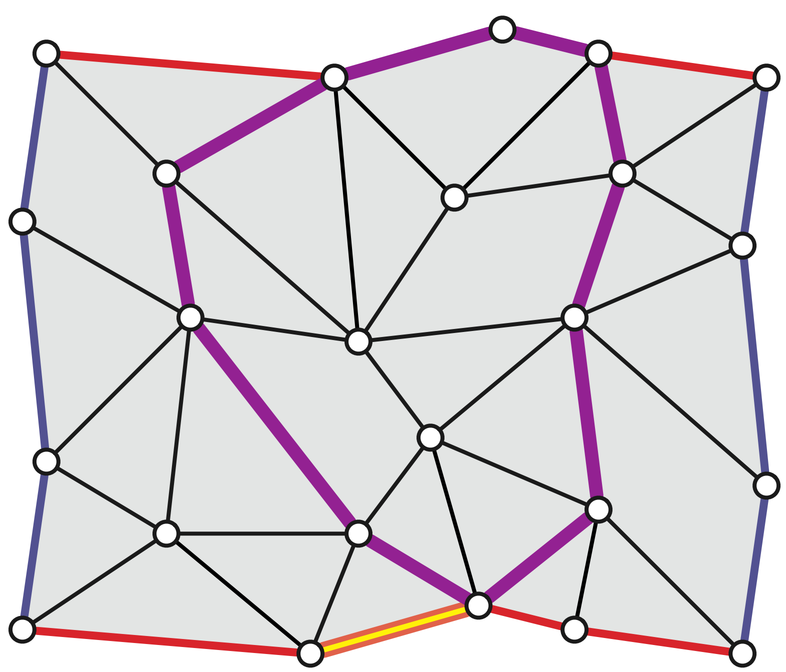
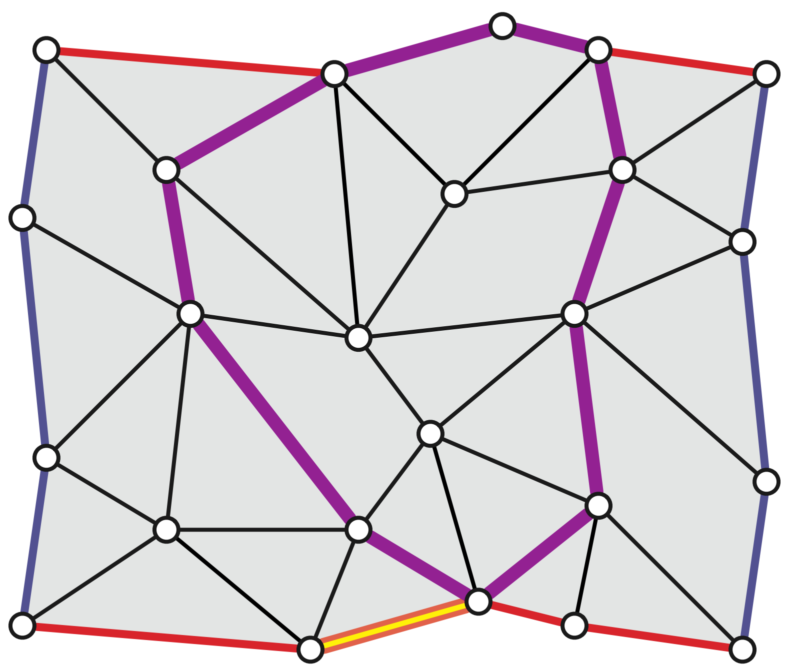


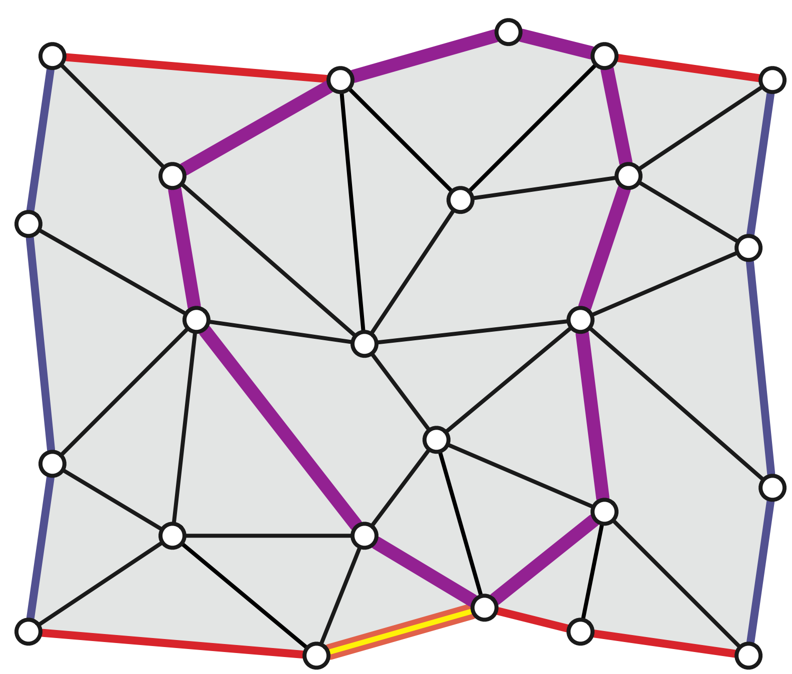
Forbidden Edge Pairs

- Pick a sliced edge e with copies e_1 and e_2
- Find min. cycle avoiding e_1 and min. cycle avoiding e_2
- Return smaller of the two results



Another Case Down

- Both cycles have at least one outside face so edges of cycle *will* separate some faces of surface graphs
- Shortest contractible cycle would have to avoid one copy anyway



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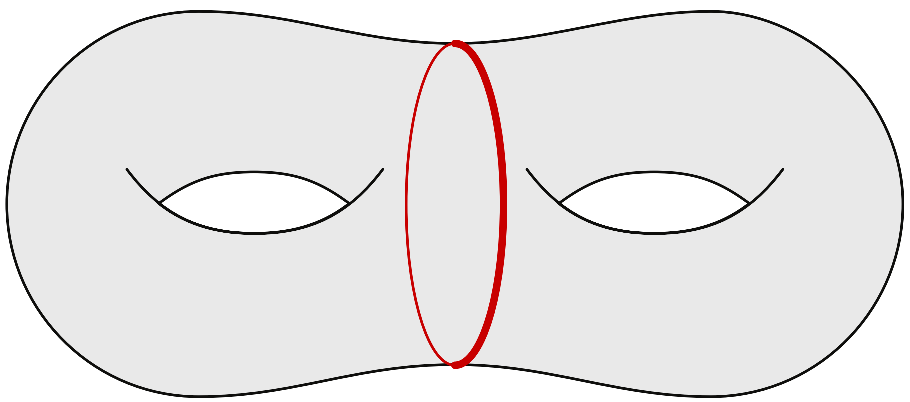
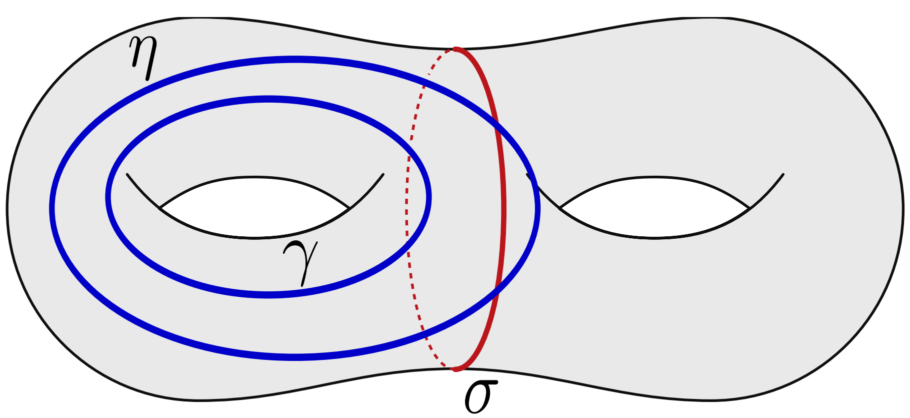


Splitting Subgraph

- Separates graph into two halves with genus
- Intuitively acts as a barrier between features of the surface

Stick to One Side

* Let γ be a non-separating closed walk not crossing the min. separating subgraph σ. The minimum even subgraph η homologous to γ lies in the same component of the σ separated surface as γ



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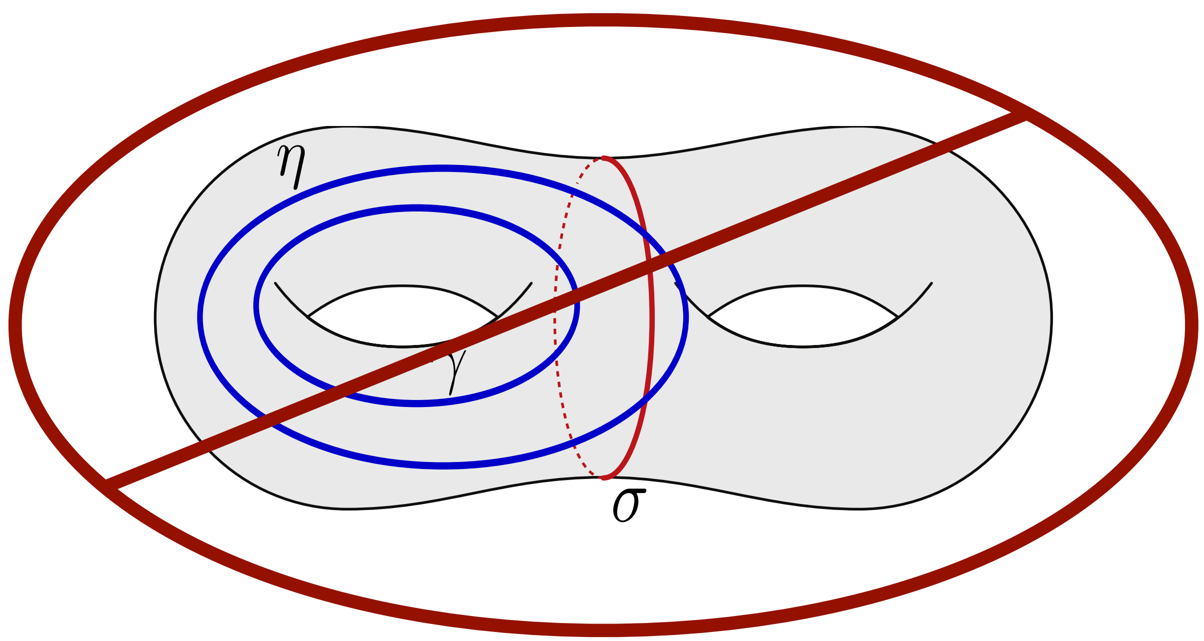
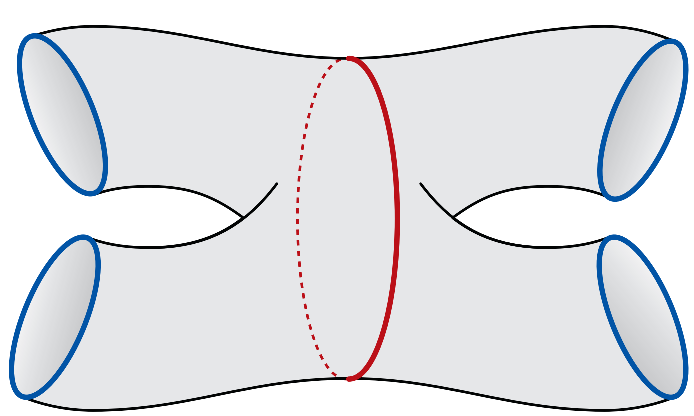


Pairs of Features

- Both sides of the cut have genus and interesting homology
- We try all $2^{O(g)}$ pairs of homology classes to find one that is separated
- For each pair, find the min. sep. subgraphs in $g^{O(g)}\, n \log\log n$ time and take one cycle from each subgraph

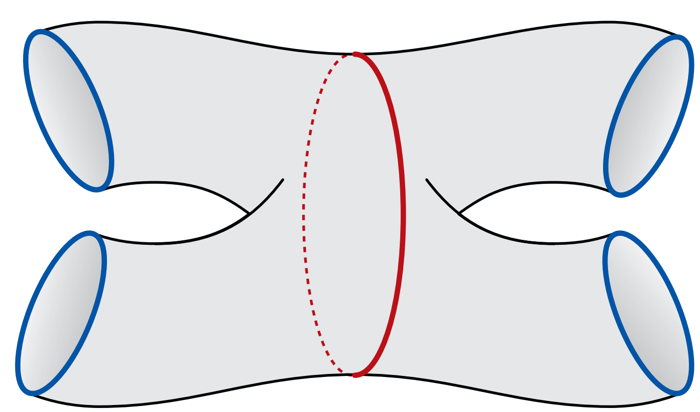
Slicing Each Side

- Suppose we know a pair of cycles α_1 and α_2 separated by the min. sep. subgraph σ
- Slice each cycle to surround σ with boundaries/pant legs



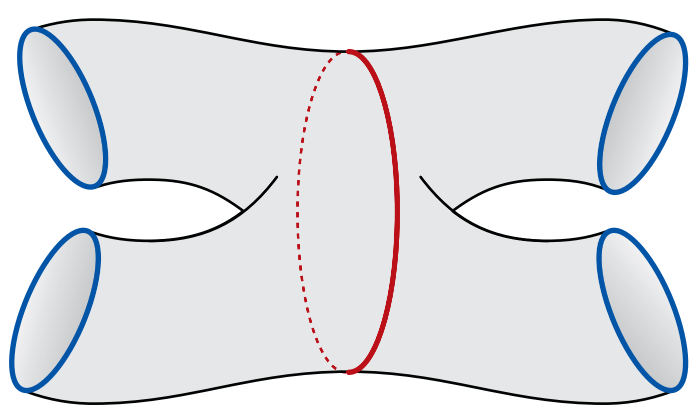
Separating the Borders

- separating subgraphs are homologous to $\alpha_1' \oplus \alpha_1''$ (the borders on the left)
- Can find $\mathbb{Z}_2$-minimal even subgraph homologous to $\alpha_1' \oplus \alpha_1''$ but may only return both halves of a sliced cycle



More Forbidden Edges

- Pick an edge e_1 on α_1 sliced into e_1' and e_1" and an edge e_2 on α_2 sliced into e_2' and e_2"
- Find $\mathbb{Z}_2$-minimal subgraphs avoiding edges e_1' and e_2', edges e_1' and e_2", *etc.* in $g^{O(g)}\, n \log \log n$ time each



Final Case Closed

- Return the smallest subgraph found
- Each subgraph separates a face next to each forbidden edge in original graph
- Shortest separating subgraph would have to avoid one copy e_1 and one copy of e_2 anyway

The Ways to Separate

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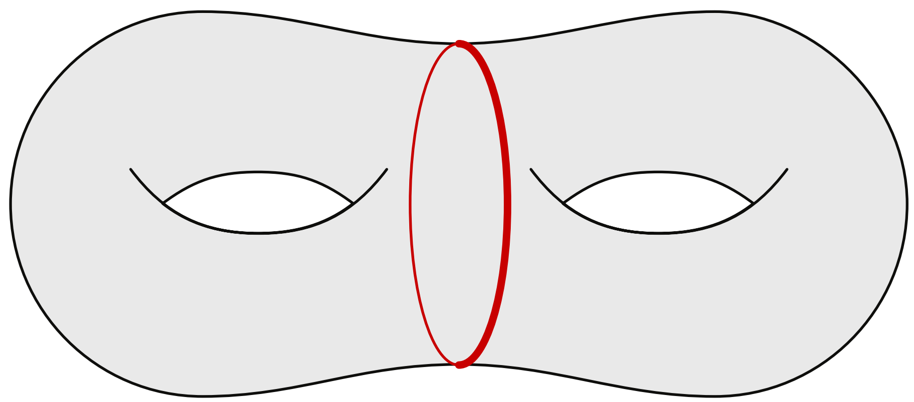

Conclusion

- Return the smallest result found for each case
- Smallest separating subgraphs and minimum cuts in $g^{O(g)} n \log \log n$ time

Open Problems

- We conjecture a $O(g^k\, n \log \log n)$ time algorithm exists for some small constant k
  - NP-hard to find arbitrary $\mathbb{Z}_2$-minimal even subgraphs
- Can we find the smallest separating simple cycle with no repeating vertices?
  - Also NP-hard, but reduction uses a polynomially complex surface