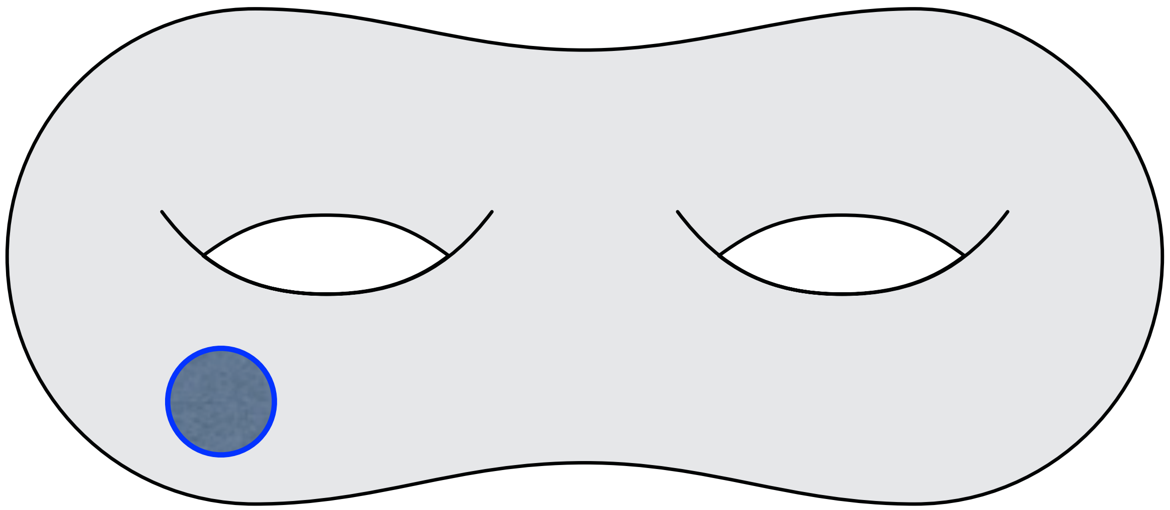
Faster Shortest Non-contractible Cycles in Directed Surface Graphs

Kyle Fox

University of Illinois at Urbana-Champaign

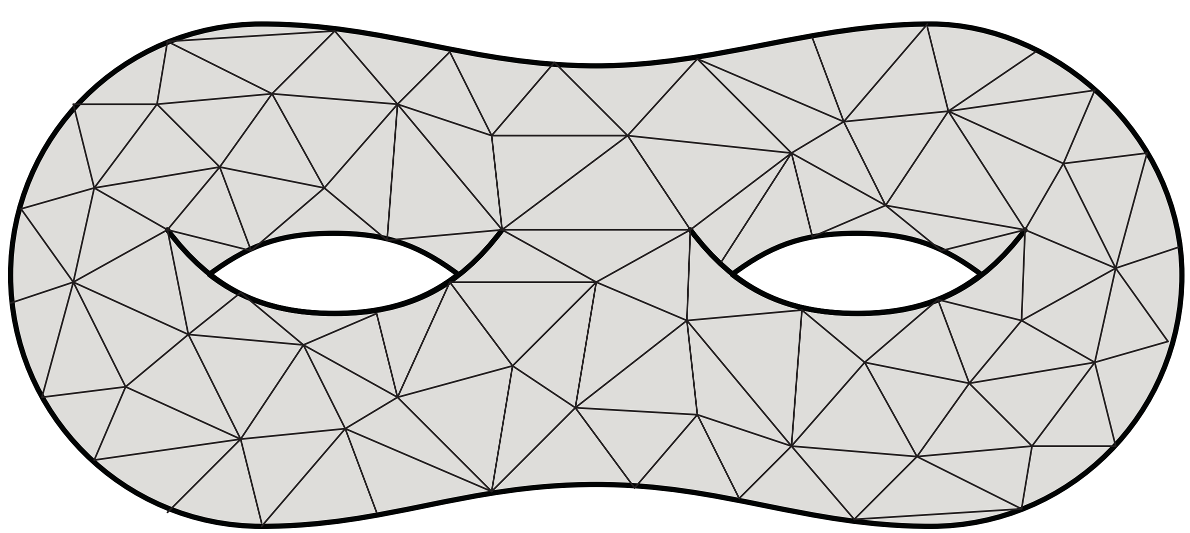
Surfaces

- 2-manifolds (with boundary)
- ***genus g***: max # of disjoint simple cycles whose compliment is connected
  = number of holes
  = number of handles attached to sphere



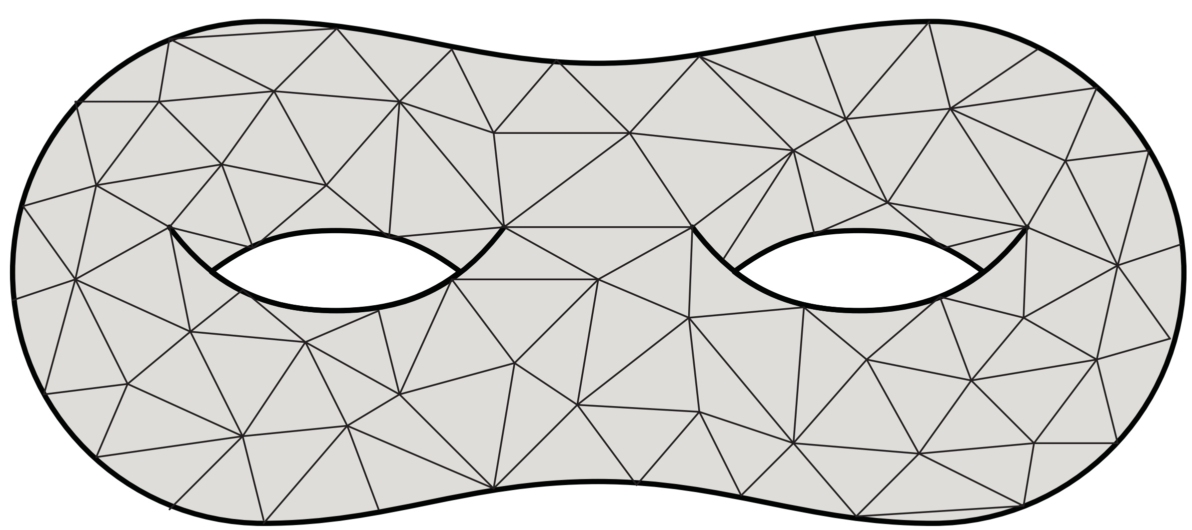
Surface Graphs

- *n* vertices as points
- *m* edges as (mostly) disjoint curves



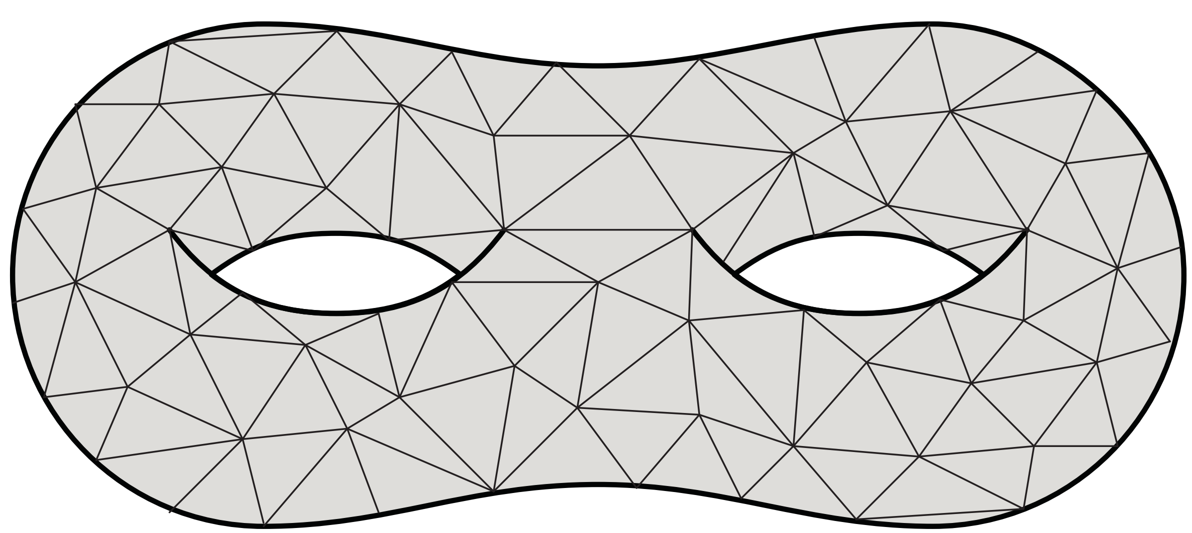
Surface Graphs

- n vertices as points
- m edges as (mostly) disjoint curves
- Assume $g = O(n)$ and $m = O(n)$



Surface Graphs

- *n* vertices as points
- *m* edges as (mostly) disjoint curves
- Assume $g = O(n)$ and $m = O(n)$
- We want to find non-trivial cycles

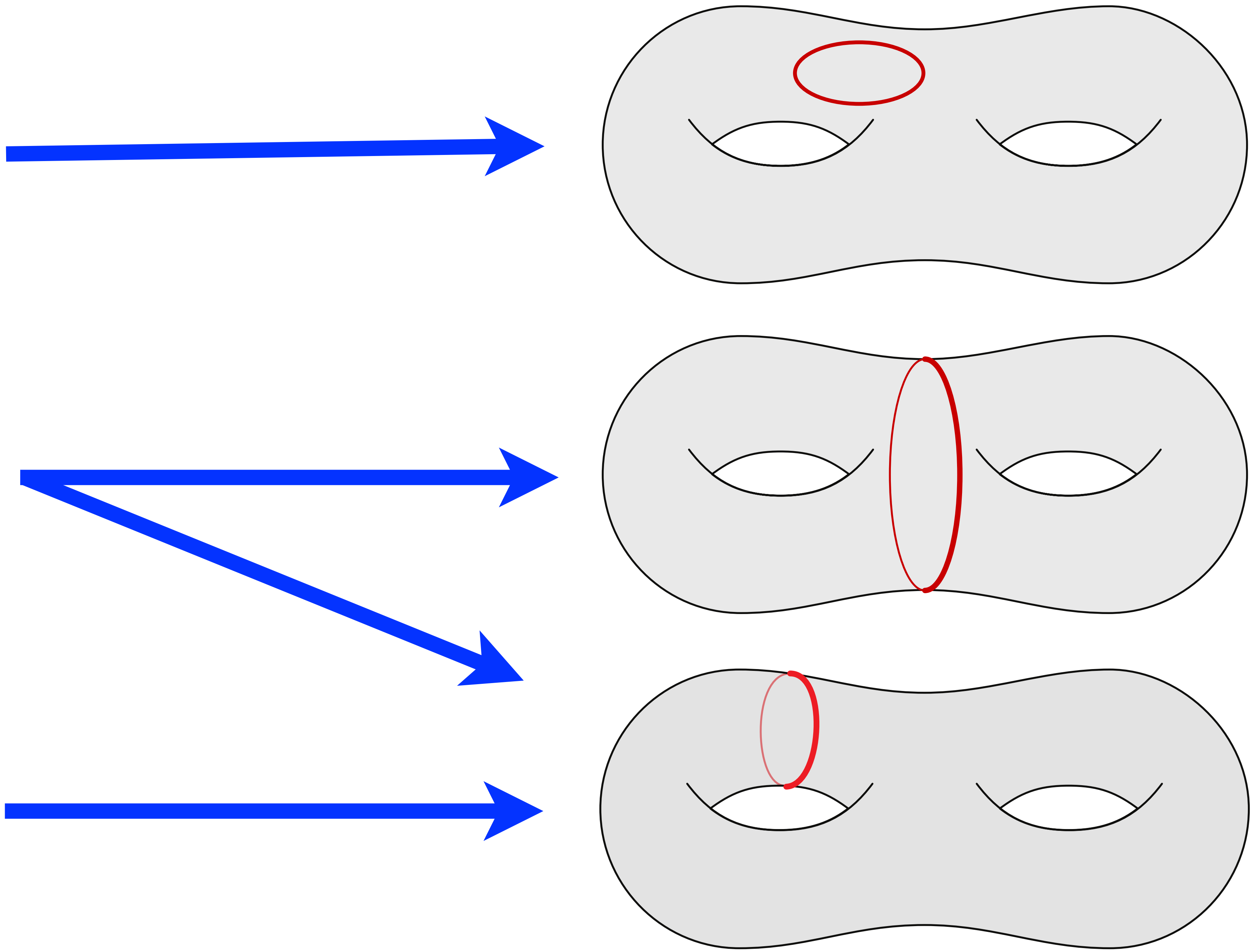


Non-trivial Cycles

Trivial ಠ_ಠ

Non-contractible

Non-separating



Finding Short Non-trivial Cycles

- Want to minimize sum of real edge lengths
- Natural question for surface embedded graphs
- Cutting along non-trivial cycles reduces the complexity of the graph
- Useful for combinatorial optimization, graphics, graph drawing, …

Results (Undirected)

Non-con.	Non-sep.	
$O(n^3)$	$O(n^3)$	[Thomassen '90]

Results (Undirected)

Non-con.	Non-sep.	
$O(n^3)$	$O(n^3)$	[Thomassen '90]
$O(n^2 \log n)$	$O(n^2 \log n)$	[Erickson, Har-peled '04]

Results (Undirected)

Non-con.	Non-sep.	
$O(n^3)$	$O(n^3)$	[Thomassen '90]
$O(n^2 \log n)$	$O(n^2 \log n)$	[Erickson, Har-peled '04]
$g^{O(g)} n^{3/2}$	$O(g^{3/2} n^{3/2} \log n + g^{5/2} n^{1/2})$	[Cabello, Mohar '07]

Results (Undirected)

Non-con.	Non-sep.	
$O(n^3)$	$O(n^3)$	[Thomassen '90]
$O(n^2 \log n)$	$O(n^2 \log n)$	[Erickson, Har-peled '04]
$g^{O(g)}\, n^{3/2}$	$O(g^{3/2}\, n^{3/2} \log n + g^{5/2}\, n^{1/2})$	[Cabello, Mohar '07]
$g^{O(g)}\, n \log n$	$g^{O(g)}\, n \log n$	[Kutz '06]

Results (Undirected)

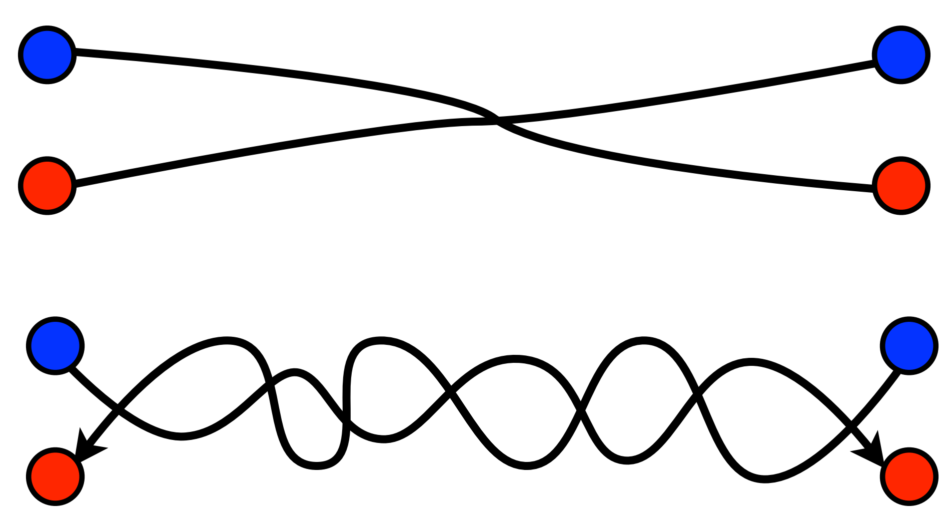
Non-con.	Non-sep.	
$O(n^3)$	$O(n^3)$	[Thomassen '90]
$O(n^2 \log n)$	$O(n^2 \log n)$	[Erickson, Har-peled '04]
$g^{O(g)}\, n^{3/2}$	$O(g^{3/2}\, n^{3/2} \log n + g^{5/2}\, n^{1/2})$	[Cabello, Mohar '07]
$g^{O(g)}\, n \log n$	$g^{O(g)}\, n \log n$	[Kutz '06]
$O(g^2\, n \log n)$	$O(g^2\, n \log n)$	[Cabello, Chambers '06; C, C, Erickson '12]

Results (Undirected)

Non-con.	Non-sep.	
$O(n^3)$	$O(n^3)$	[Thomassen '90]
$O(n^2 \log n)$	$O(n^2 \log n)$	[Erickson, Har-peled '04]
$g^{O(g)} n^{3/2}$	$O(g^{3/2} n^{3/2} \log n + g^{5/2} n^{1/2})$	[Cabello, Mohar '07]
$g^{O(g)} n \log n$	$g^{O(g)} n \log n$	[Kutz '06]
$O(g^2 n \log n)$	$O(g^2 n \log n)$	[Cabello, Chambers '06; C, C, Erickson '12]
$g^{O(g)} n \log \log n$	$g^{O(g)} n \log \log n$	[Italiano, *et al.* '11]

Undirected Edges are Kind

- Walks have the same length as their reversals
- Shortest paths cross at most once
- Neither holds in general for directed graphs



Results (Directed)

Non-con.	Non-sep.	
$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	[Cabello, Colin de Verdière, Lazarus '10]

Results (Directed)

Non-con.	Non-sep.	
$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	[Cabello, Colin de Verdière, Lazarus '10]
	$2^{O(g)} n \log n$	[Erickson, Nayyeri '11]

Results (Directed)

Non-con.	Non-sep.	
$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	[Cabello, Colin de Verdière, Lazarus '10]
	$2^{O(g)} n \log n$	[Erickson, Nayyeri '11]
$g^{O(g)} n \log n$	$O(g^2 n \log n)$	[Erickson '11]

Results (Directed)

Non-con.	Non-sep.	
$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	[Cabello, Colin de Verdière, Lazarus '10]
	$2^{O(g)} n \log n$	[Erickson, Nayyeri '11]
$g^{O(g)} n \log n$	$O(g^2 n \log n)$	[Erickson '11]
$O(g^3 n \log n)$		[F '12]

Results (Directed)

Non-con.	Non-sep.	
$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	$O(n^2 \log n)$ and $O(g^{1/2} n^{3/2} \log n)$	[Cabello, Colin de Verdière, Lazarus '10]
	$2^{O(g)} n \log n$	[Erickson, Nayyeri '11]
$g^{O(g)} n \log n$	$O(g^2 n \log n)$	[Erickson '11]
$\mathbf{O(g^3 n \log n)}$		**[F '12]**

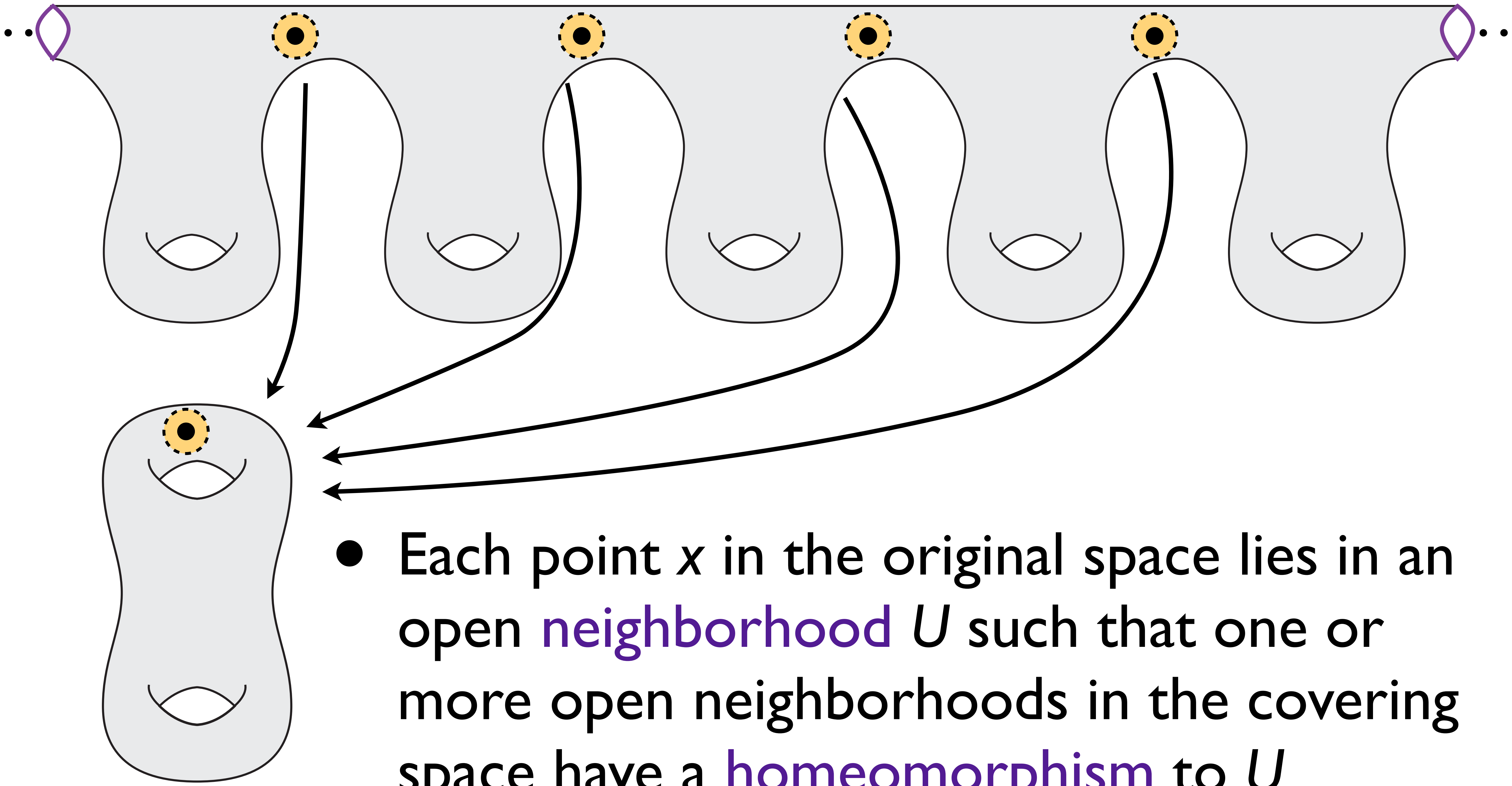
Assumptions

- If the shortest non-contractible cycle is separating, we can use the algorithm of Erickson
- Presentation assumes the cycle is separating and the surface has exactly one boundary cycle

Main Ideas

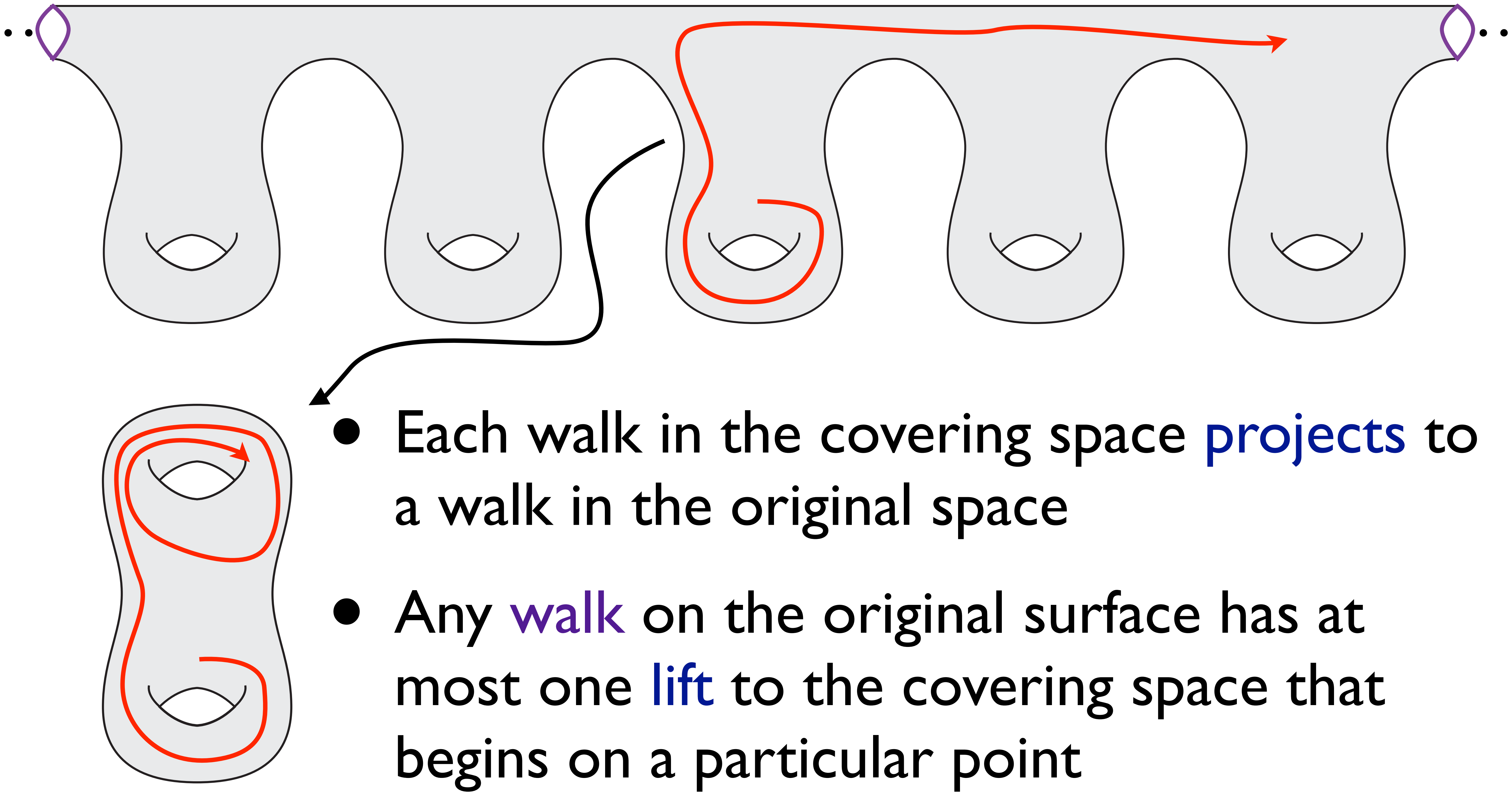
- Lift the graph to one of O(g) copies of a covering space
- The shortest non-contractible cycle is non-null-homologous in one of the lifted copies

Covering Spaces



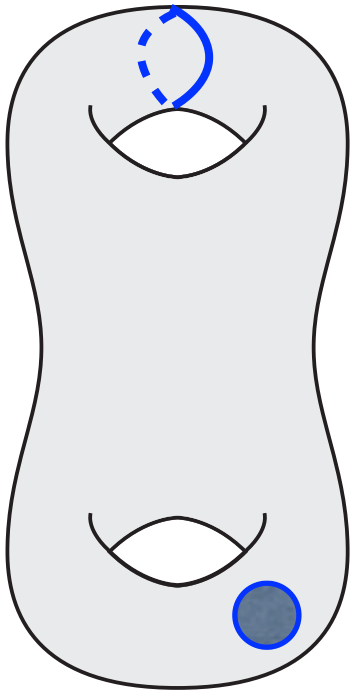
- Each point *x* in the original space lies in an open neighborhood *U* such that one or more open neighborhoods in the covering space have a homeomorphism to *U*

Covering Spaces



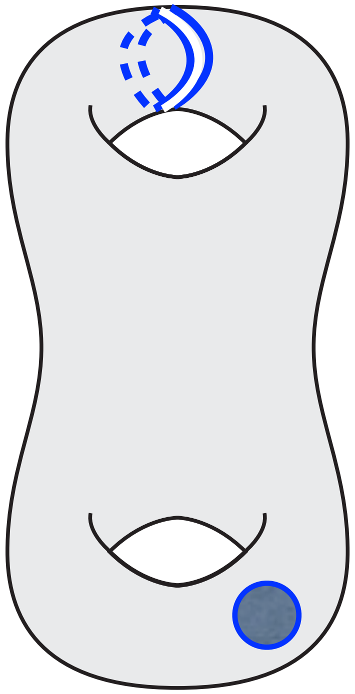
- Each walk in the covering space projects to a walk in the original space
- Any walk on the original surface has at most one lift to the covering space that begins on a particular point

Infinite Cyclic Cover



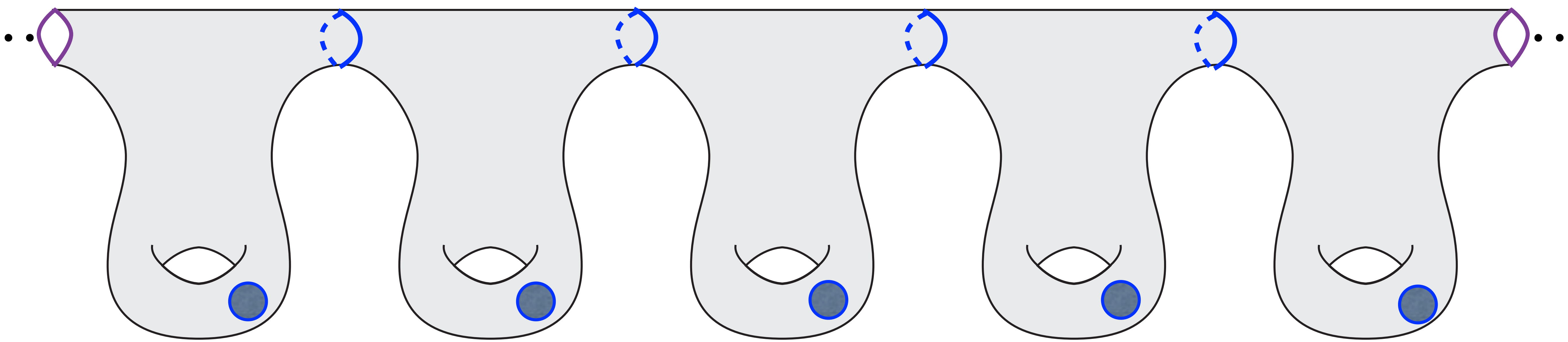
- Let λ be any non-separating cycle

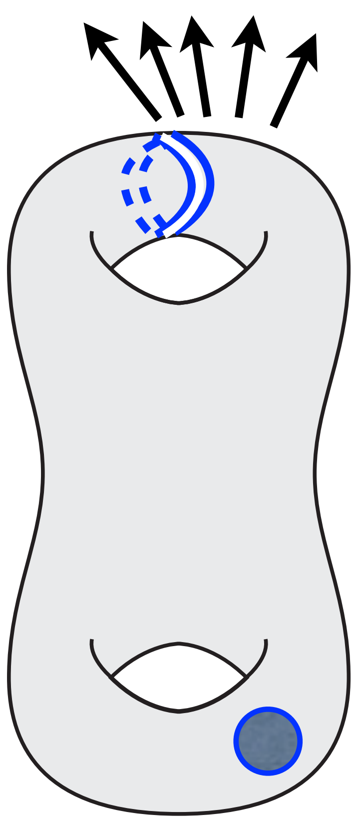
Infinite Cyclic Cover



- Cut the surface along λ

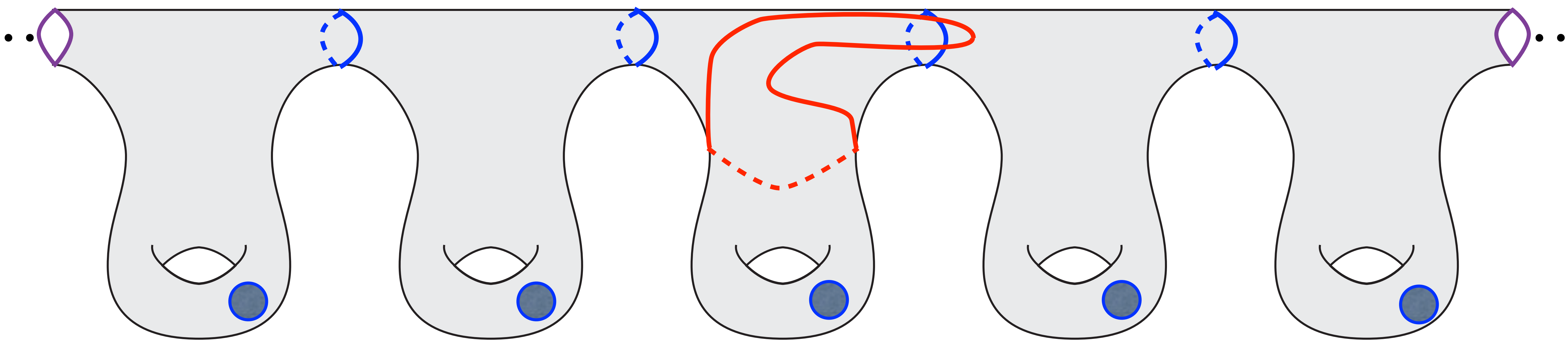
Infinite Cyclic Cover

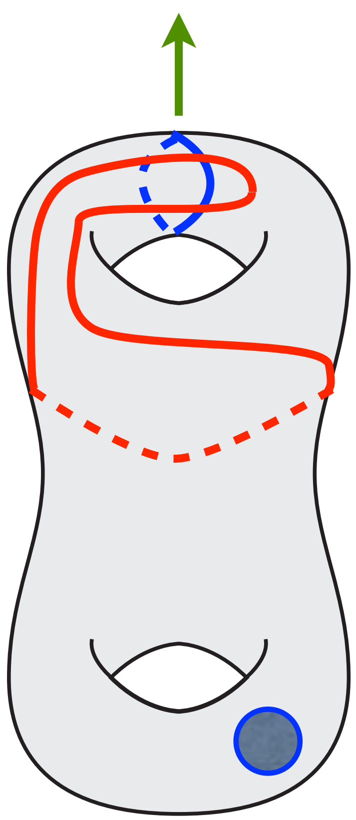




- Cut the surface along λ
- Glue an infinite number of copies together along λ

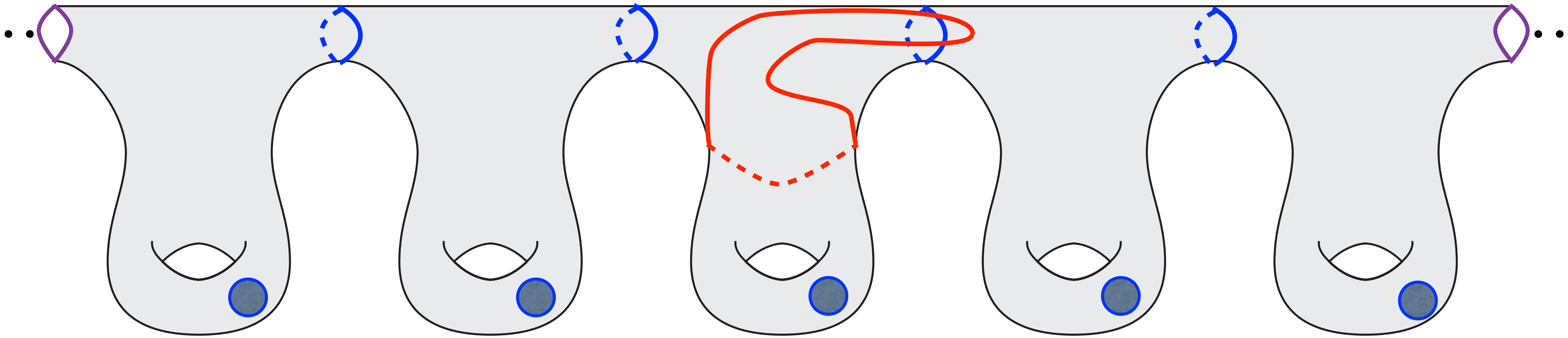
Cycles in the Cover

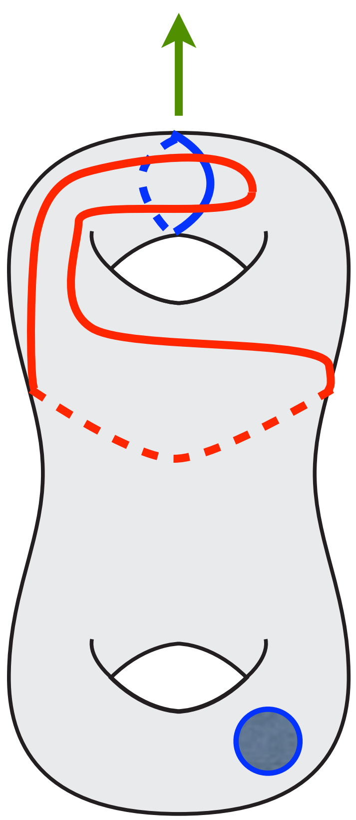




- A cycle γ on the original surface lifts to a cycle if and only if it crosses λ left to right the same number of times as it crosses right to left
- Any separating cycle lifts to a cycle

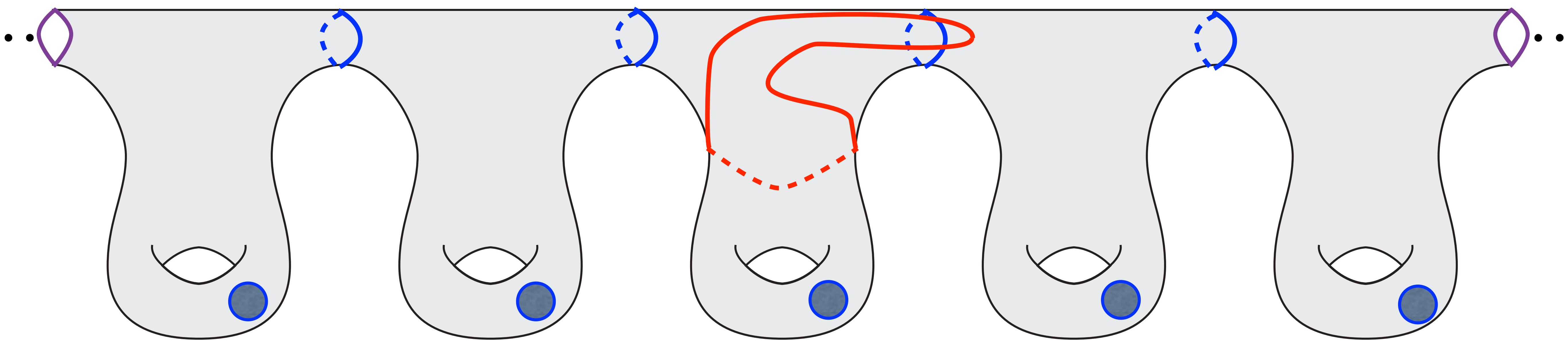
Cycles in the Cover

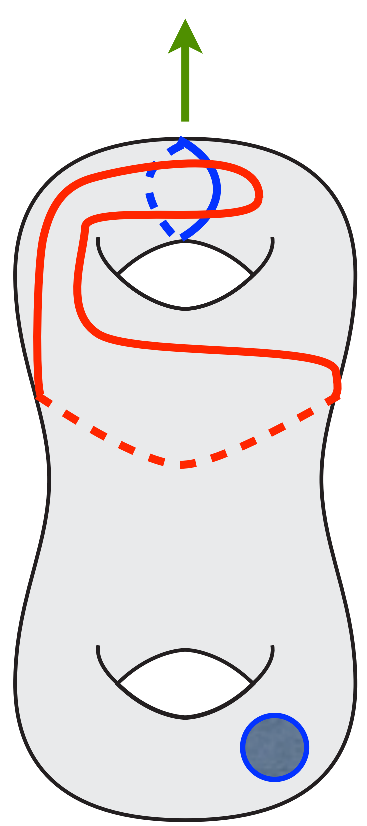




- The shortest non-contractible cycle lifts to a cycle

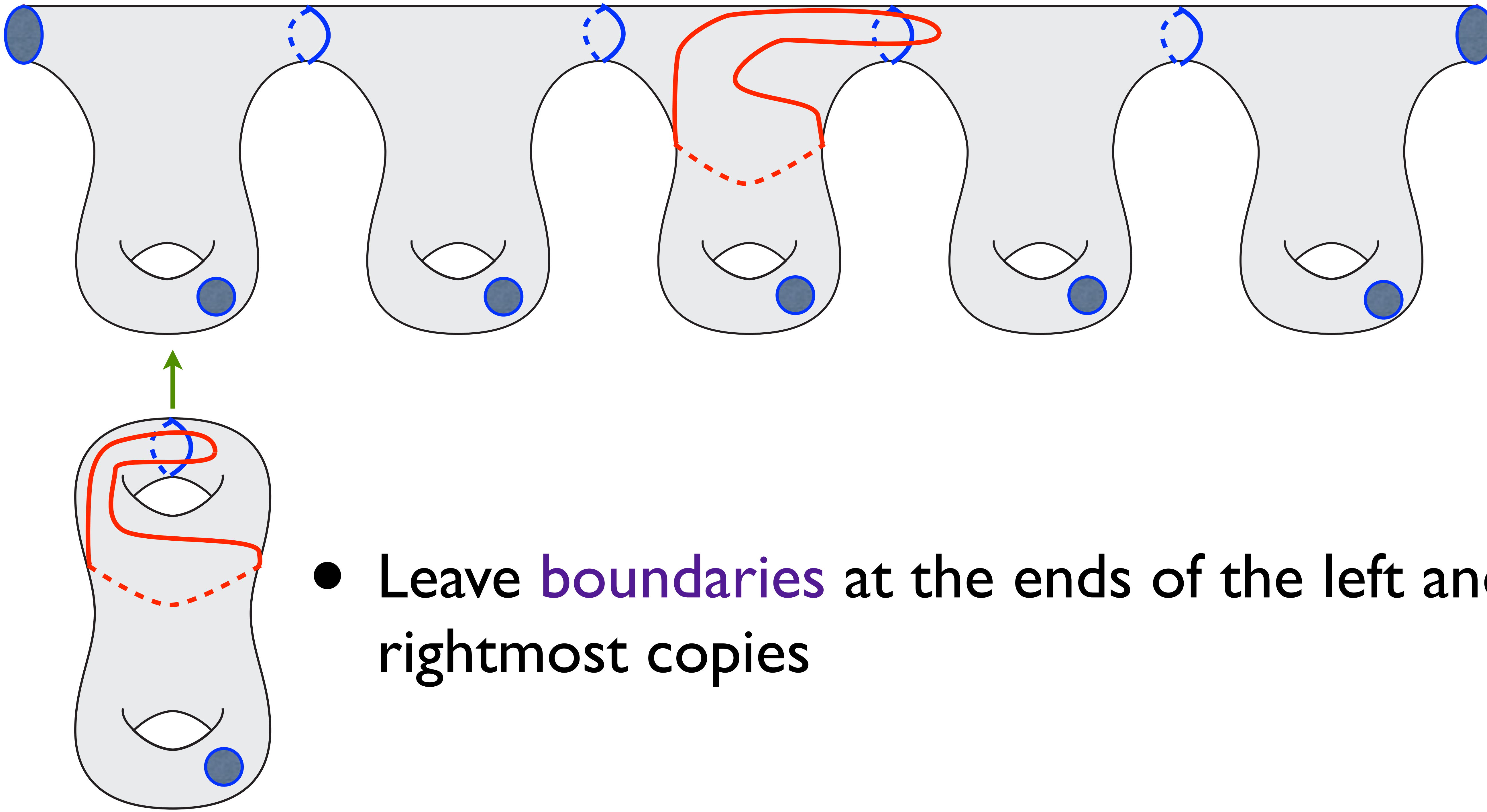
Path Intersections





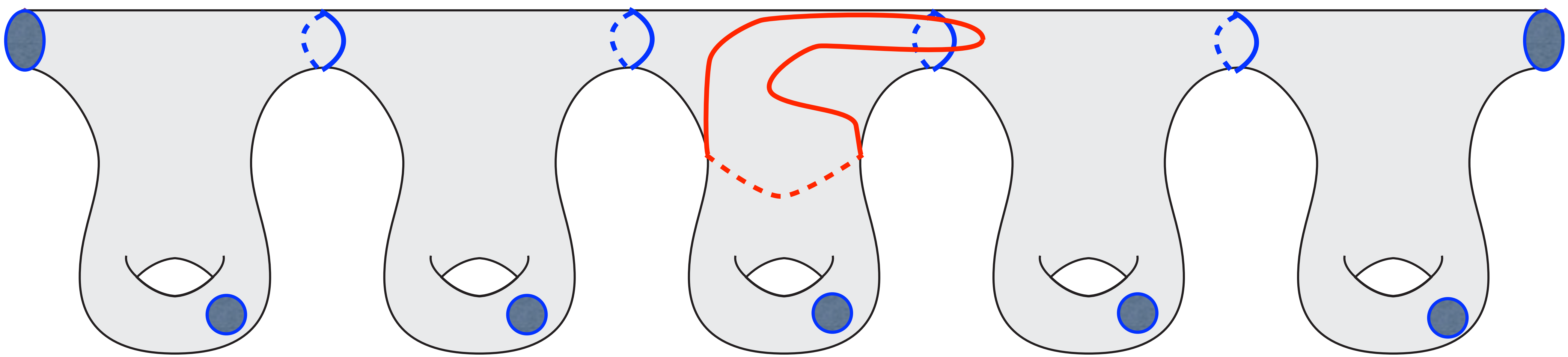
- The shortest non-contractible cycle intersects at most 2 lifts of any shortest path [Erickson '11]
- We only need 5 copies of the original surface in the cover if we cut along a cycle made from shortest paths

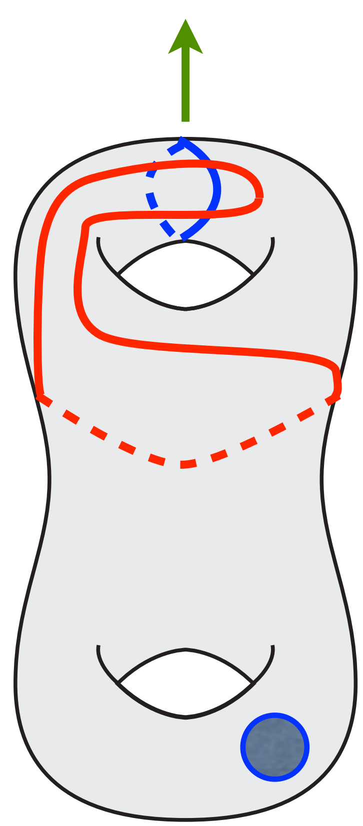
Path Intersections



- Leave boundaries at the ends of the left and rightmost copies

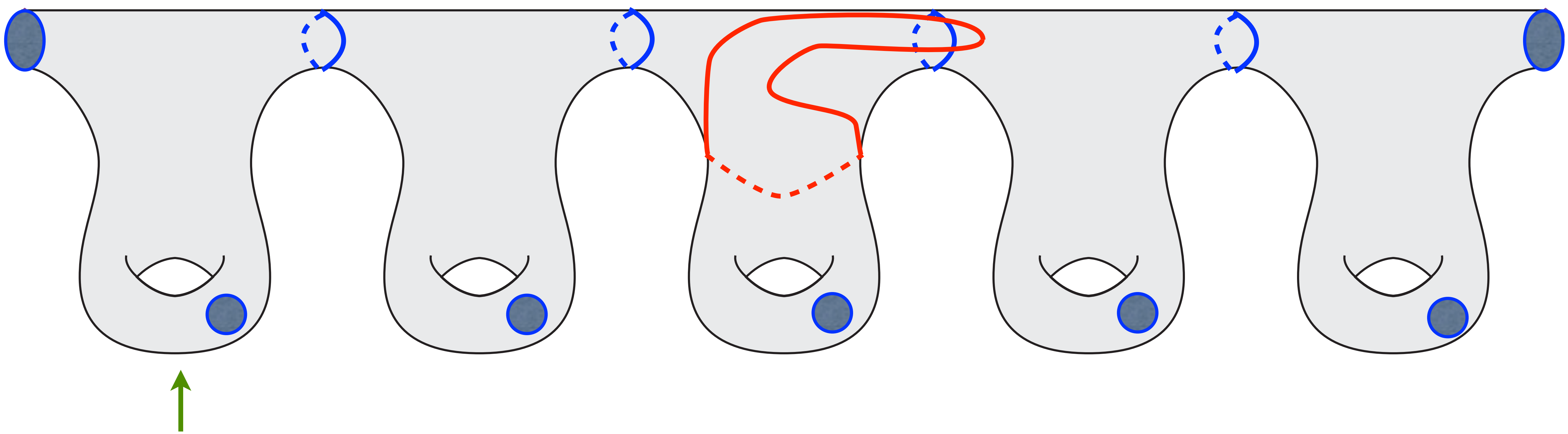
Non-contractible Lift

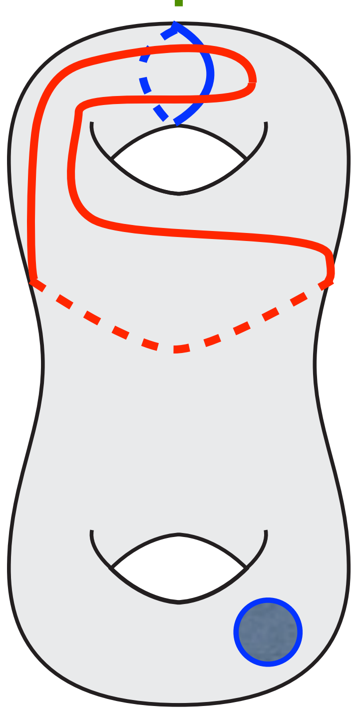




- A cycle γ' in the cover projects to a non-contractible cycle if and only if γ' is non-contractible

Non-contractible Lift





- The shortest non-contractible cycle in the original surface is the shortest non-contractible cycle in the cover

Recap

- Many non-separating cycles can be used to create the subset of the infinite cyclic cover

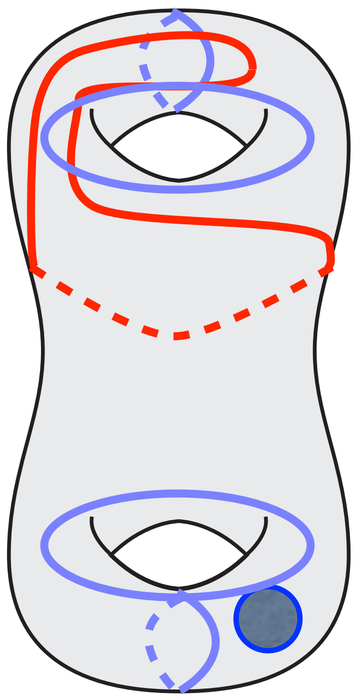
Recap

- Many non-separating cycles can be used to create the subset of the infinite cyclic cover
- Suffices to find the shortest non-contractible cycle in any subset of the cover

Recap

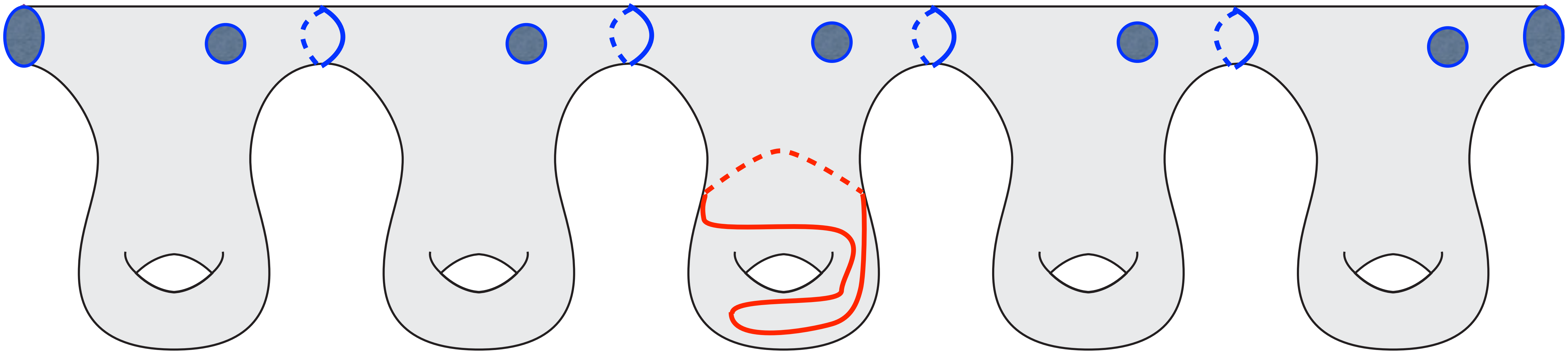
- Many non-separating cycles can be used to create the subset of the infinite cyclic cover
- Suffices to find the shortest non-contractible cycle in any subset of the cover
- But the genus increased!

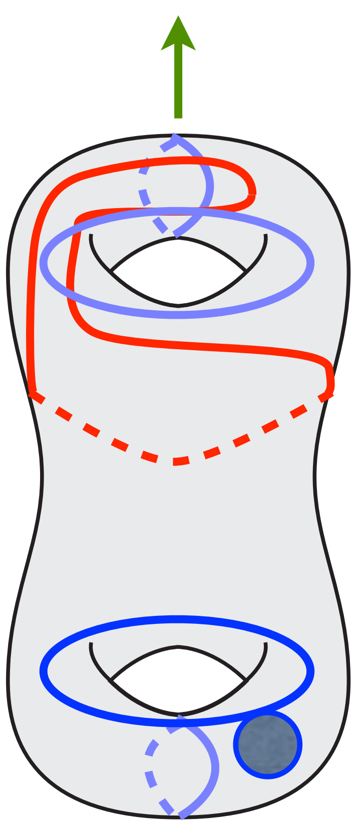
Separating Boundary



- Compute a system of 2g non-separating cycles from shortest paths in $O(n \log n + g\, n)$ time (a homology basis)

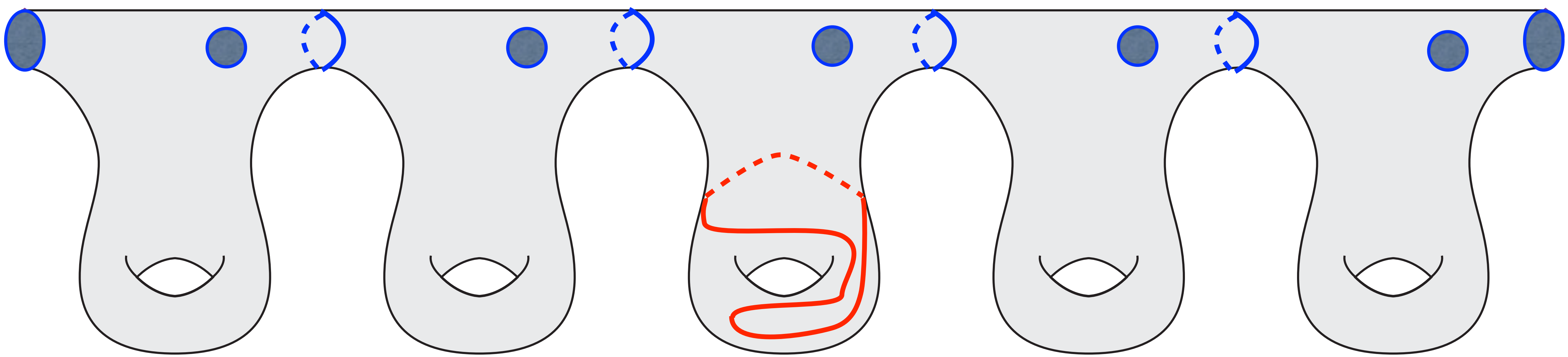
Separating Boundary

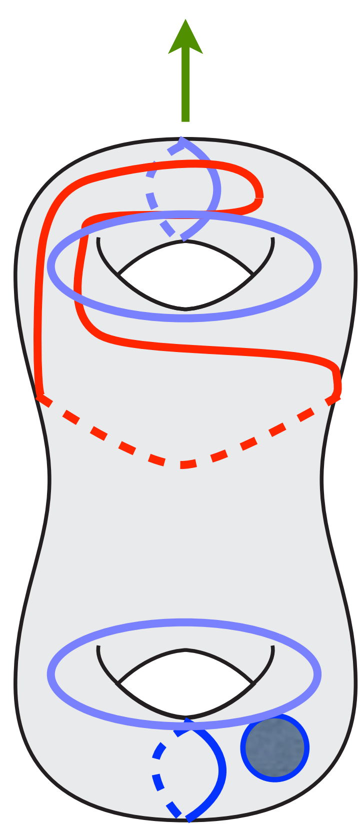




- Compute a system of 2g non-separating cycles from shortest paths in $O(n \log n + g\, n)$ time (a homology basis)

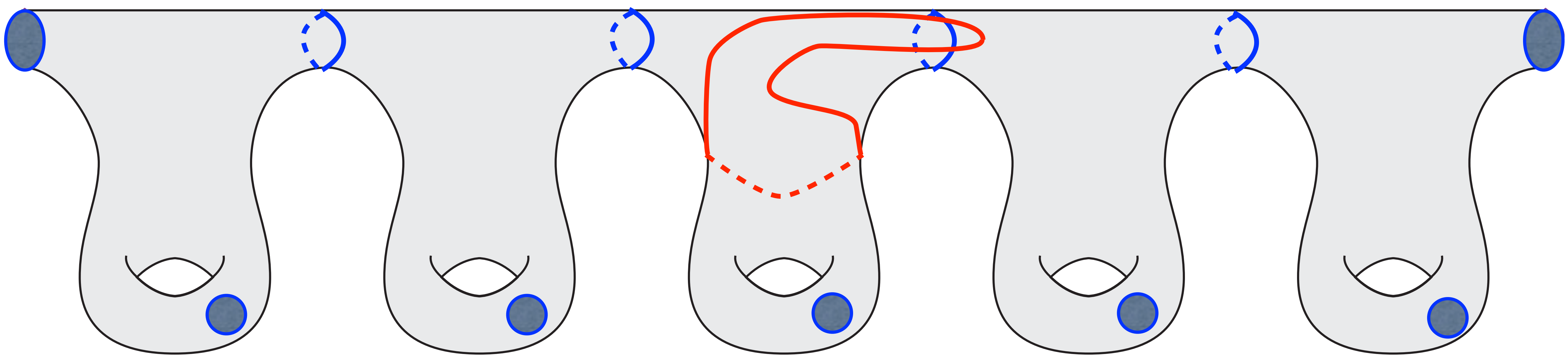
Separating Boundary

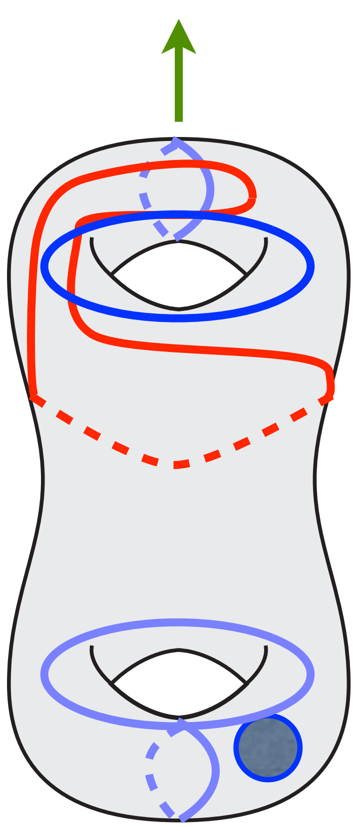




- Compute a system of 2g non-separating cycles from shortest paths in $O(n \log n + g\,n)$ time (a homology basis)

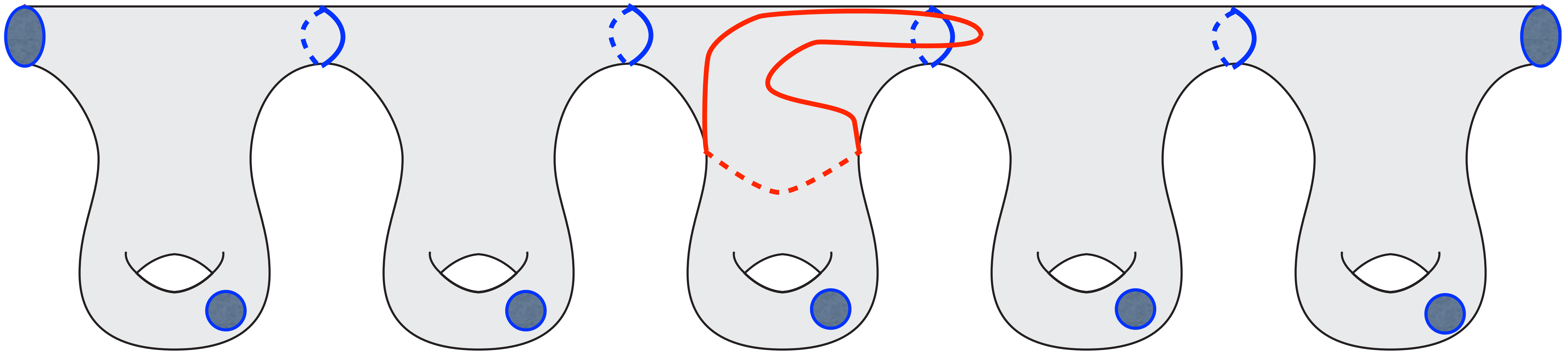
Separating Boundary

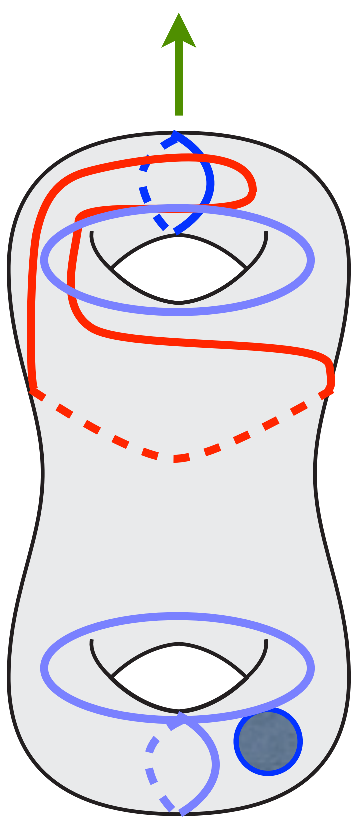




- Compute a system of 2g non-separating cycles from shortest paths in $O(n \log n + g\, n)$ time (a homology basis)

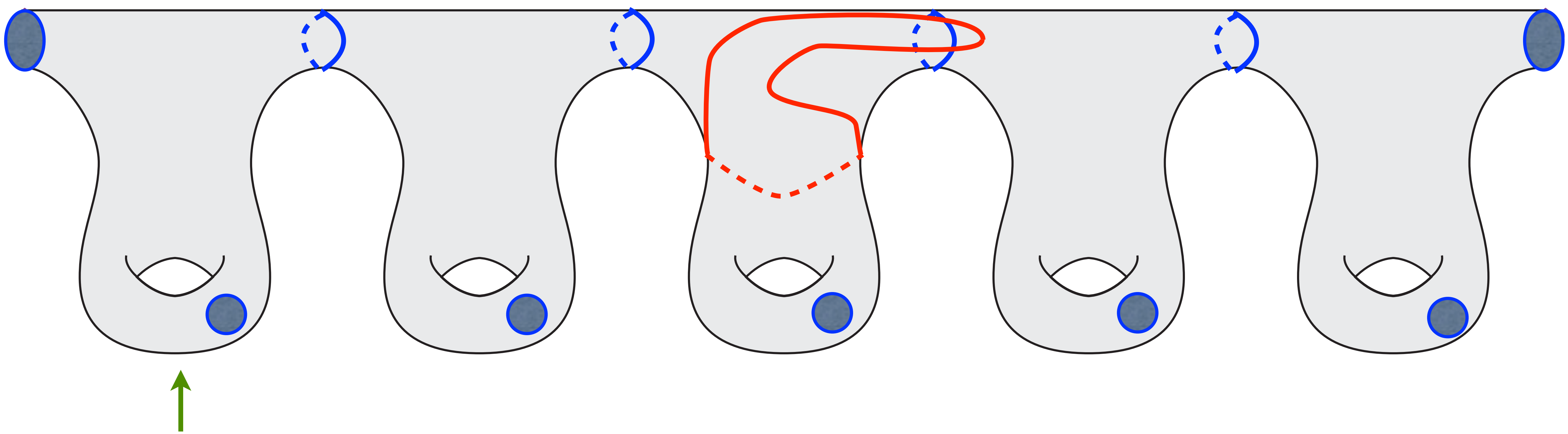
Separating Boundary

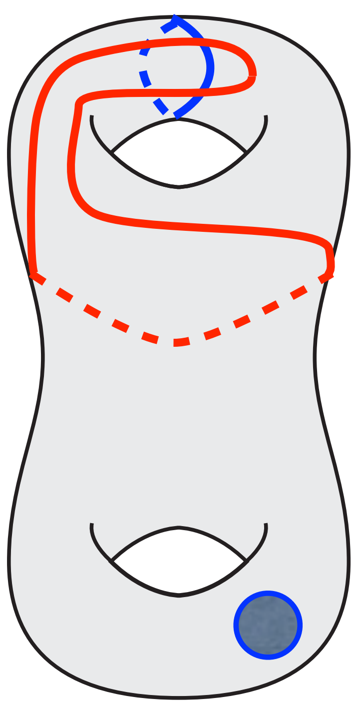




- At least one of the non-separating cycles yields a useful subset of the cover

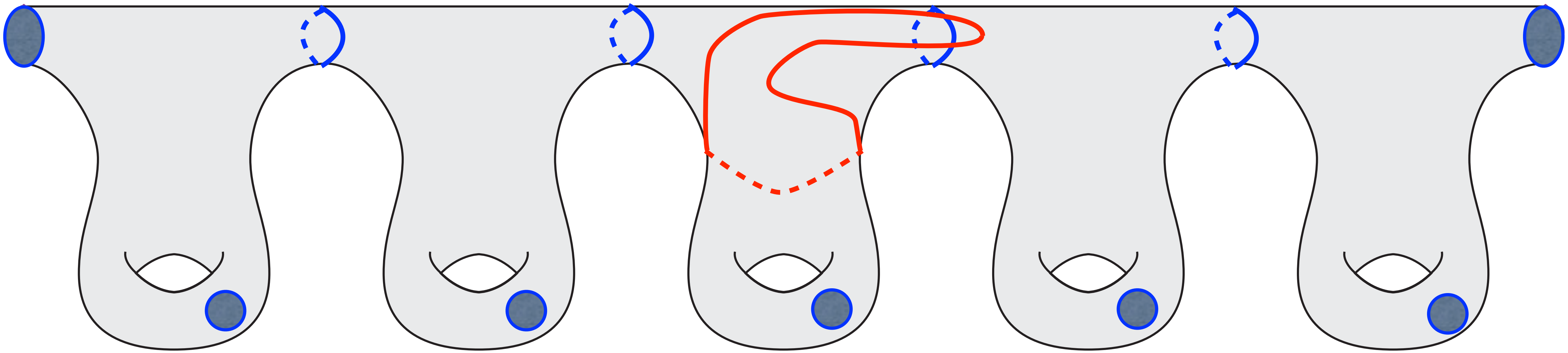
Separating Boundary

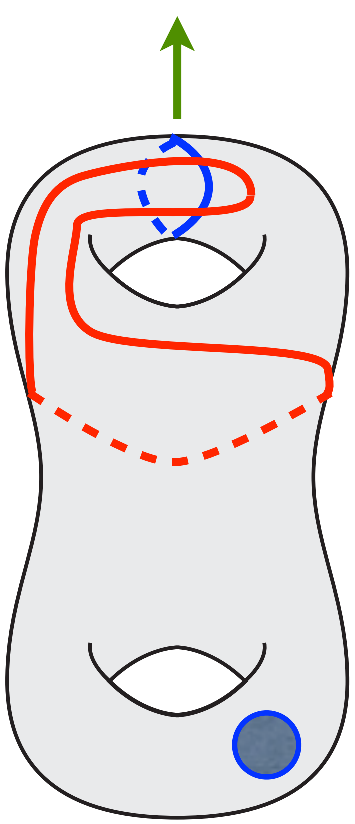




- The shortest non-contractible cycle lifts to become non-separating or to separate a pair of boundary
- The lift is non-null-homologous

Search the Covers





- In the cover, find the shortest cycle that is non-null-homologous using (a modified version of) Erickson's algorithm [Erickson '11; **F '12**]

Running Time

- Can search for short cycles in $O(g^2\, n \log n)$ time per covering space
- $O(g^3\, n \log n)$ time spent searching $2g$ covers

In Closing

- Gave an algorithm for computing shortest non-contractible cycles in directed surface graphs
- $O(g^3 n \log n)$ time – first algorithm with near-linear dependency on n and sub-exponential dependency on g

Thank you