Shortest Non-trivial Cycles in Directed and Undirected Surface Graphs

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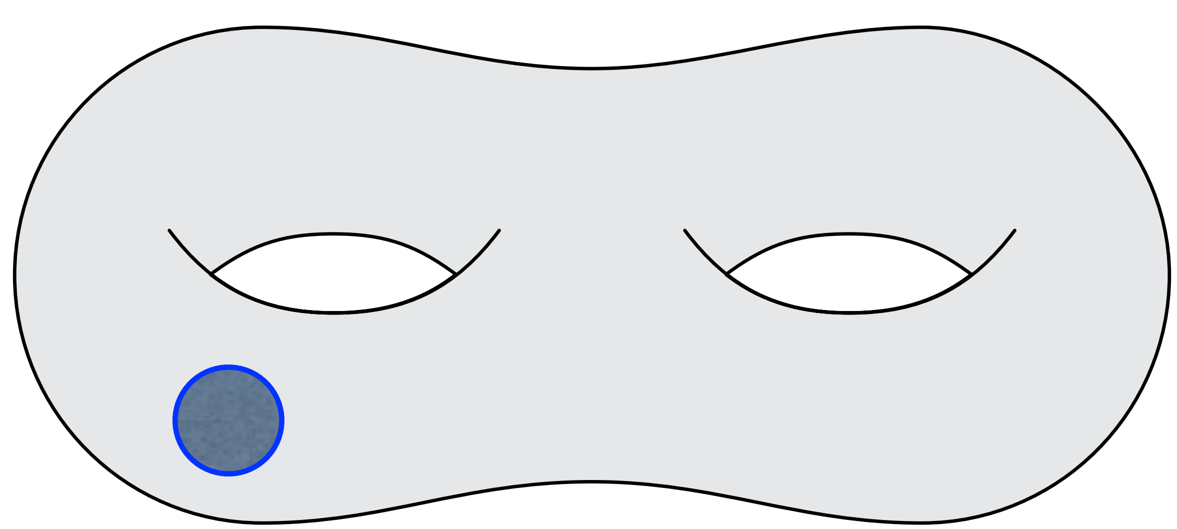
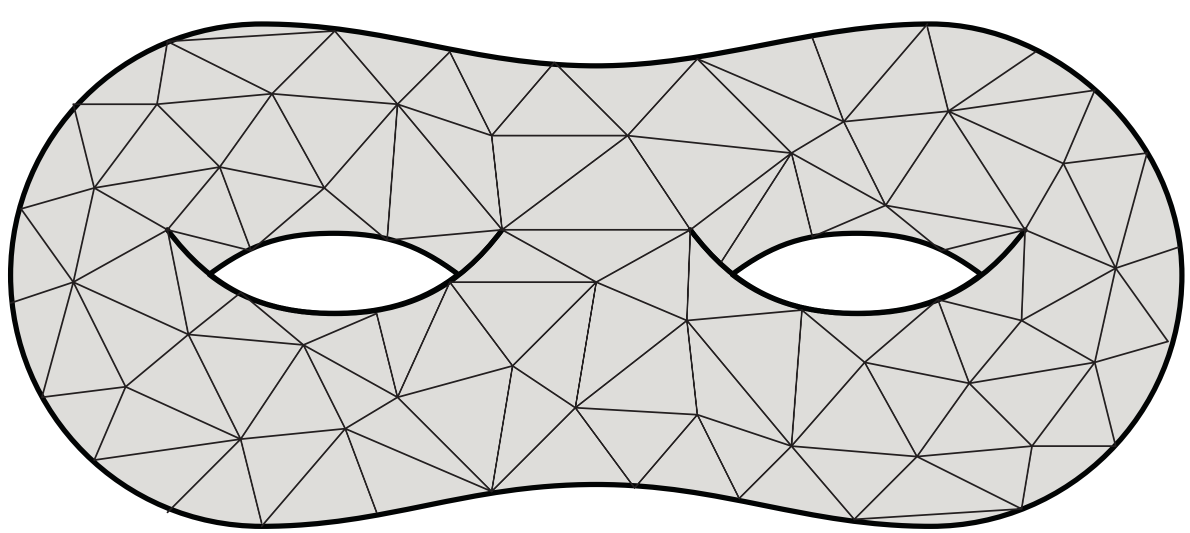
Surfaces

- 2-manifolds (with boundary)
- ***genus g***: max # of disjoint simple cycles whose compliment is connected
  = number of holes
  = number of handles attached to sphere



Surface Graphs

- *n* vertices as points
- *m* edges as (mostly) disjoint curves



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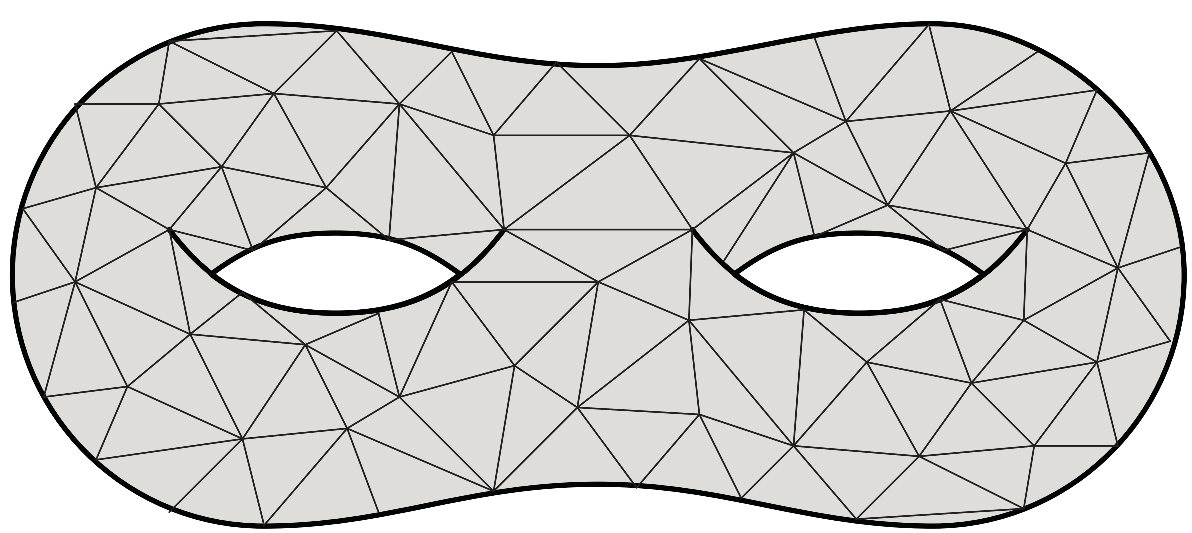

Surface Graphs

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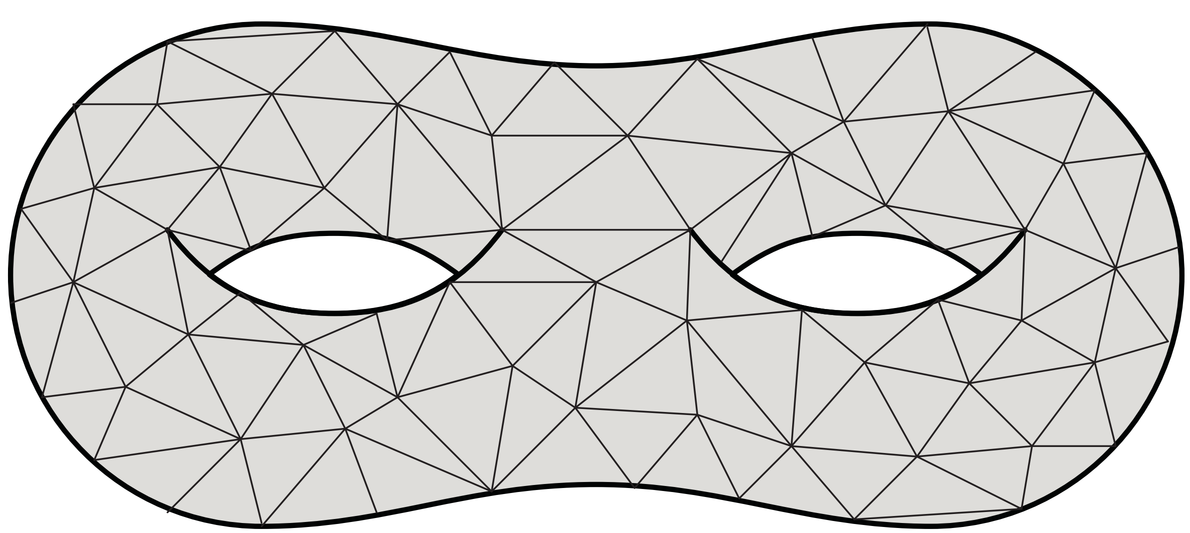
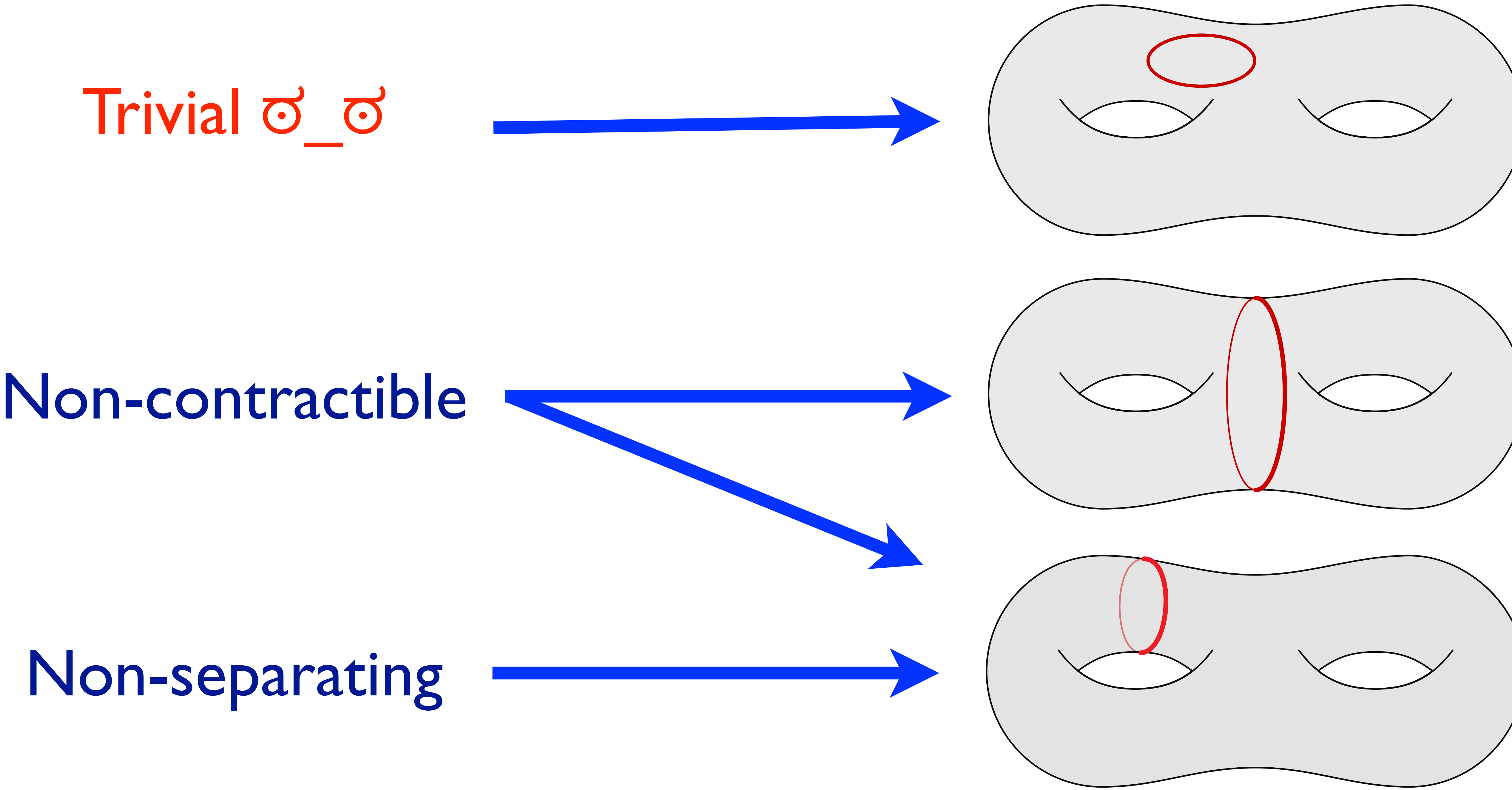
- We want to find non-trivial cycles



Non-trivial Cycles



Finding Short Non-trivial Cycles

- Want to minimize sum of real edge lengths (not geodesics)
- Natural question for surface embedded graphs
- Cutting along non-trivial cycles reduces the complexity of the graph
- Useful for combinatorial optimization, graphics, graph drawing, …

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Non-con.	Non-sep.	
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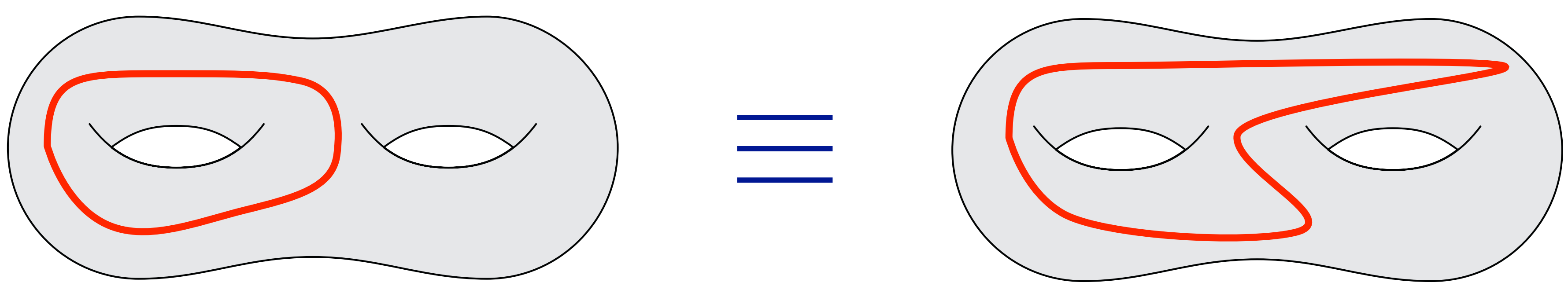
New Undirected Results

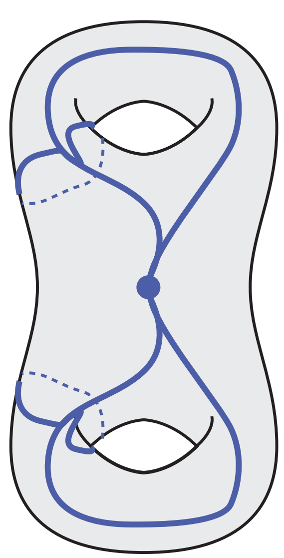
- Based on algorithm of Kutz ['06] and Italiano *et al.* ['11]
- Two cycles are homotopic if one can be continuously deformed to the other

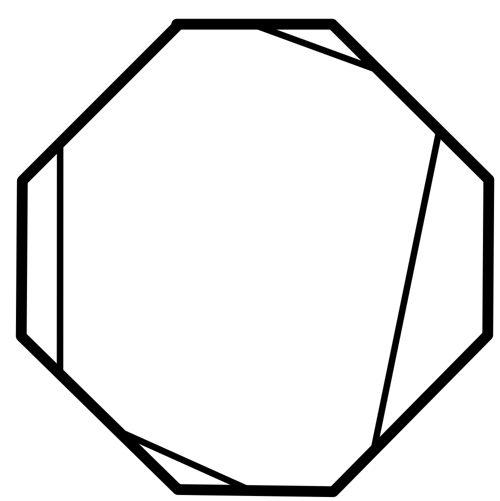


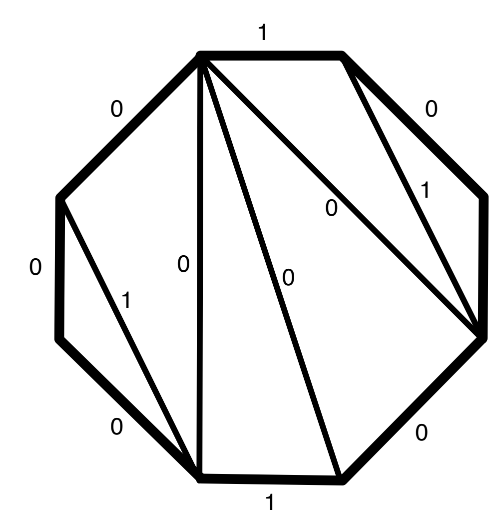
- Prior algorithms compute shortest cycles in $g^{O(g)}$ homotopy classes

New Undirected Results

- New algorithm reduces number of homotopy classes to $2^{O(g)}$
- Classes chosen based on triangulations of the polygonal schema [Chambers *et al.* '08; Chambers, Erickson, Nayyeri '09]

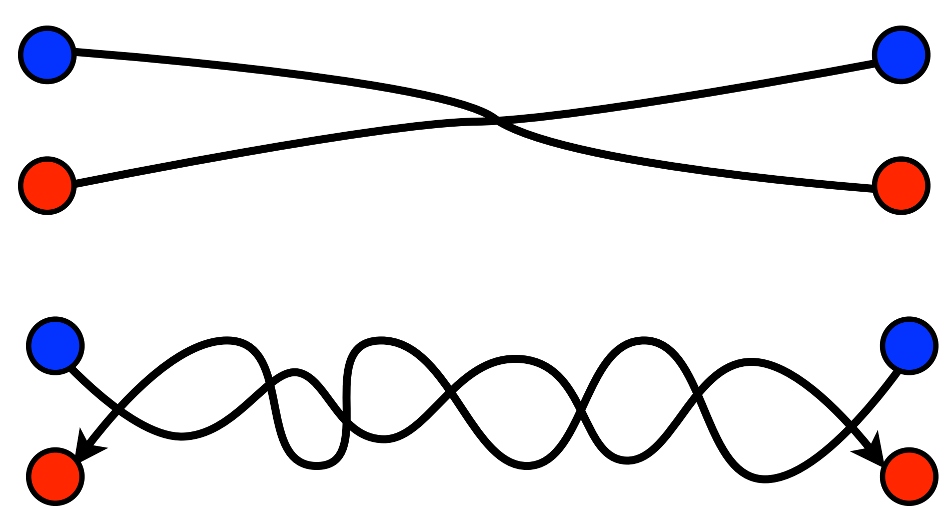






Undirected Edges are Kind

- Walks have the same length as their reversals
- Shortest paths cross at most once
- Neither holds in general for directed graphs



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$\mathbf{O(g^3 n \log n)}$		**[F '12]**

Assumptions for New Result

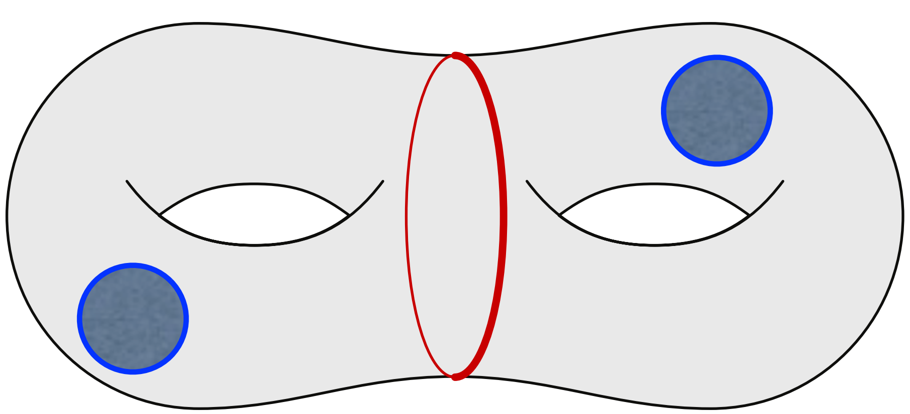
- If the shortest non-contractible cycle is separating, we can use the algorithm of Erickson
- Presentation assumes the cycle is separating and the surface has exactly one boundary cycle

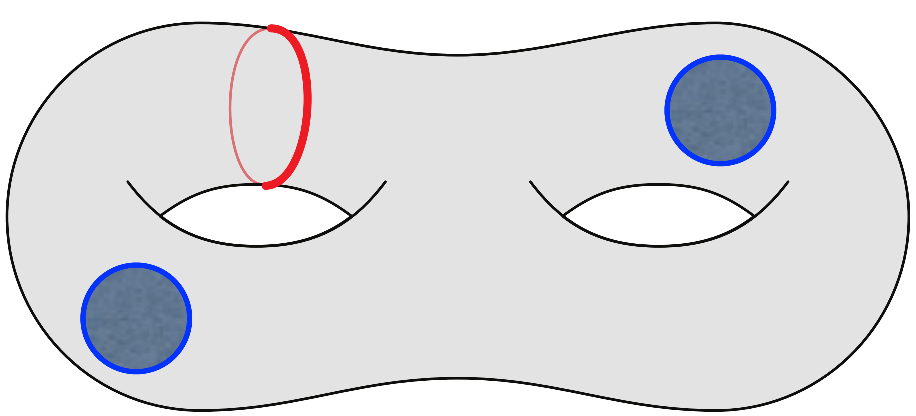
Main Ideas

- Lift the graph to one of O(g) copies of a covering space
- The shortest non-contractible cycle is non-null-homologous in one of the lifted copies

Non-null-homologous Cycles

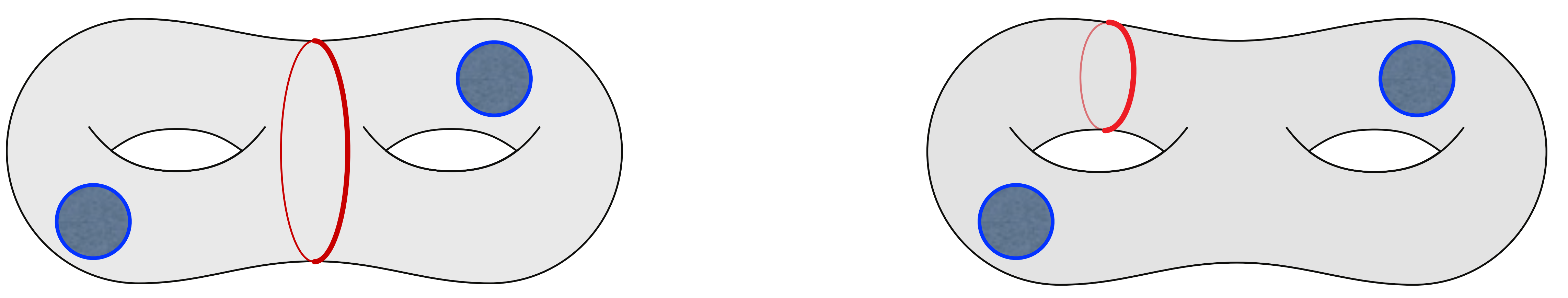
- Either non-separating or separate boundary components
- Are all non-contractible



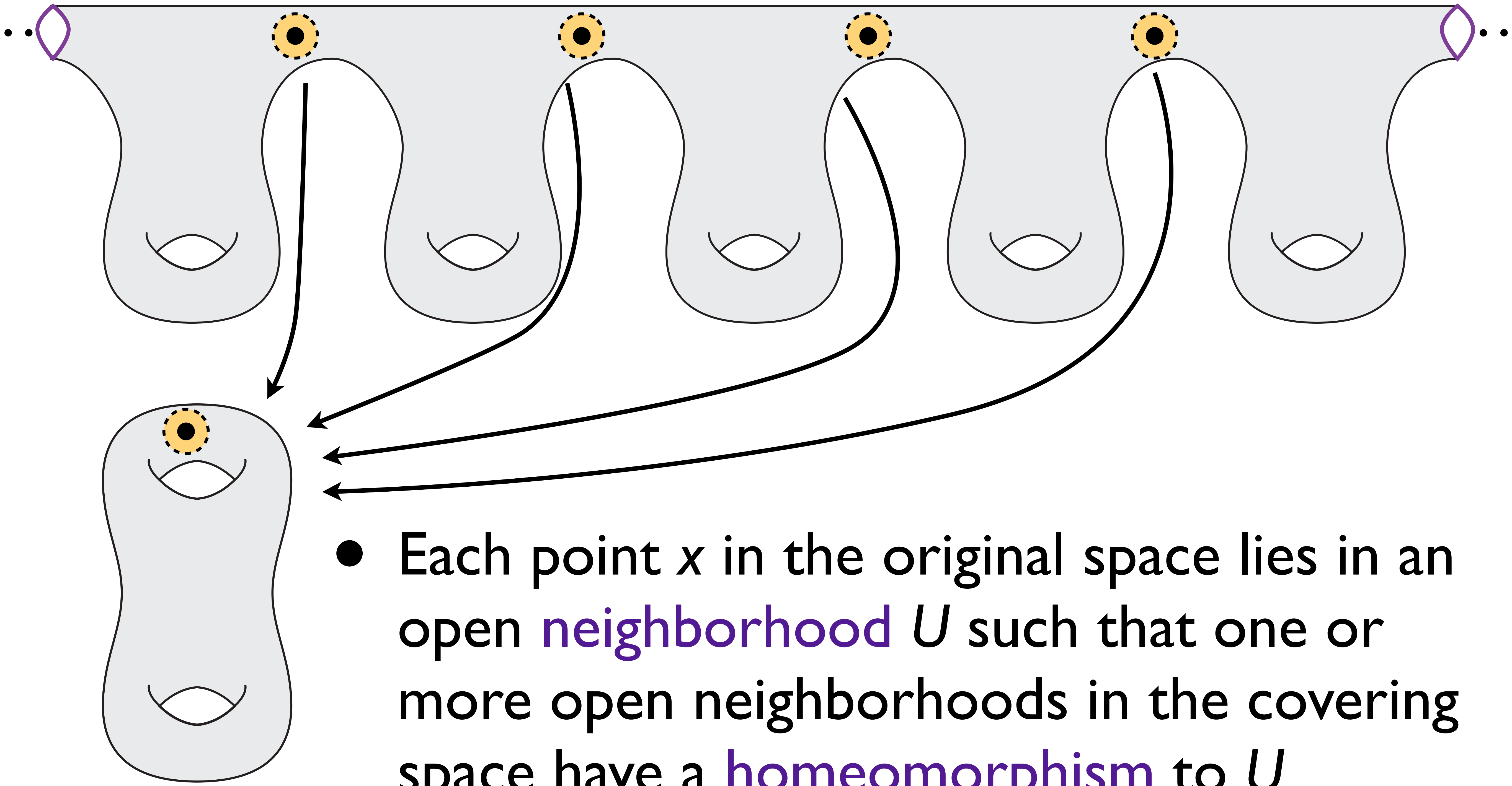


Non-null-homologous Cycles

- **Bonus Result:** Shortest non-null-homologous cycles computable as quickly as shortest non-separating cycles
- $2^{O(g)}\, n \log \log n$ time in undirected graphs
- $O(g^2\, n \log n)$ time in directed graphs



Covering Spaces



- Each point *x* in the original space lies in an open neighborhood *U* such that one or more open neighborhoods in the covering space have a homeomorphism to *U*

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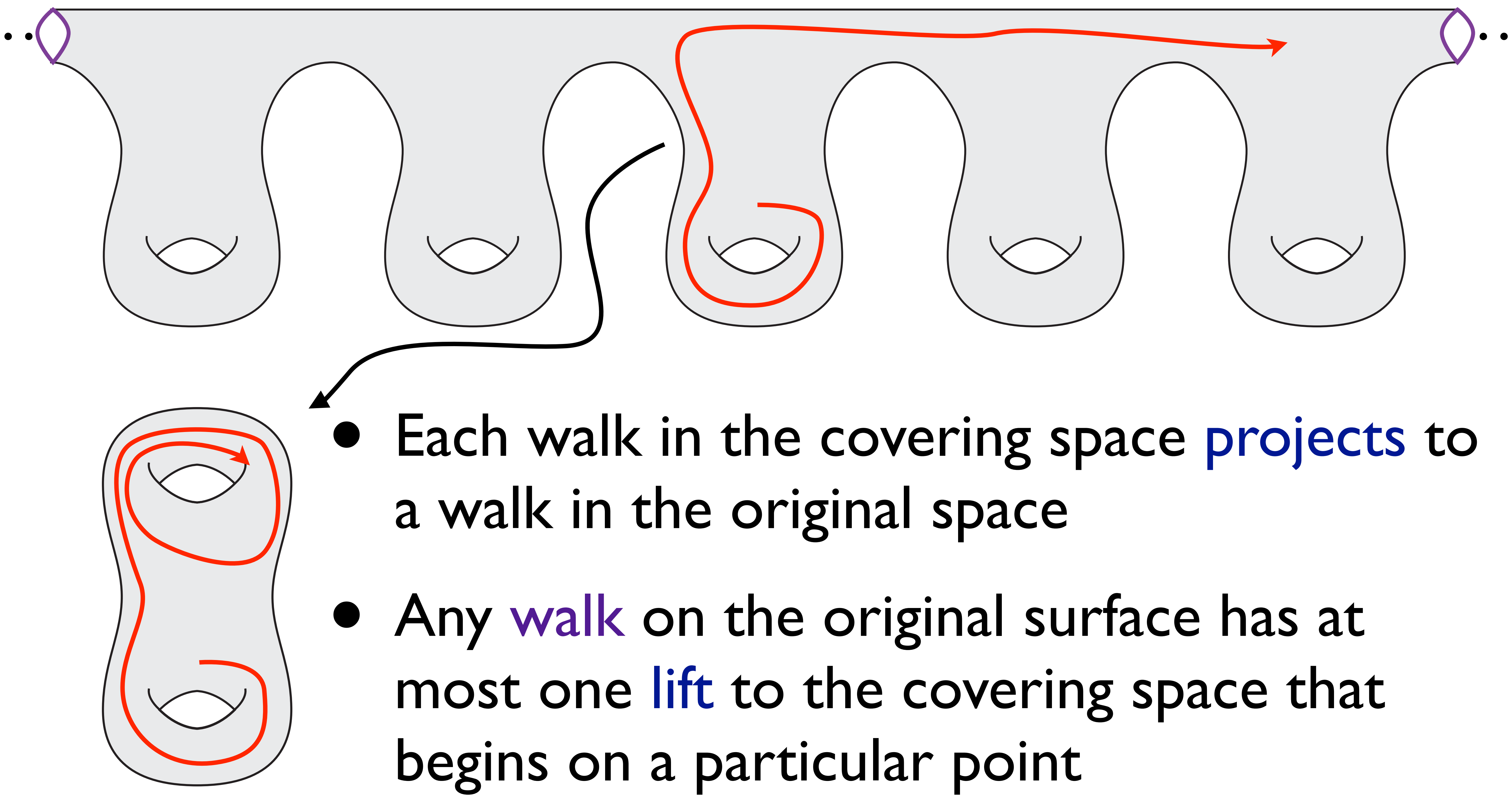
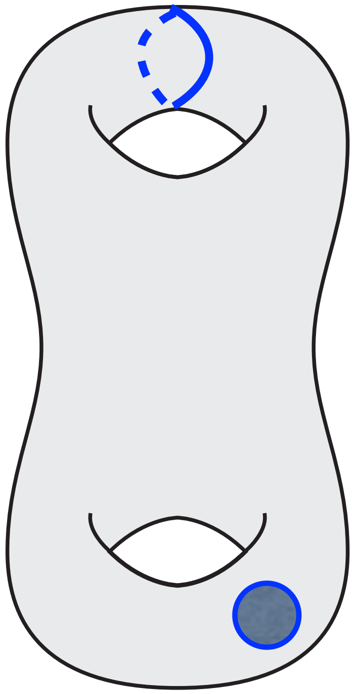


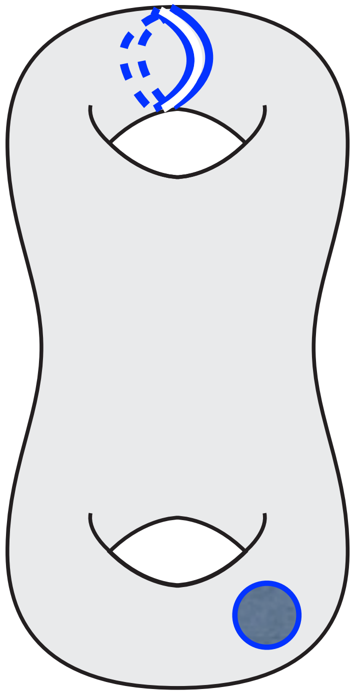
- Each walk in the covering space projects to a walk in the original space
- Any walk on the original surface has at most one lift to the covering space that begins on a particular point

Infinite Cyclic Cover



- Let λ be *any* non-separating cycle

Infinite Cyclic Cover



- Cut the surface along λ

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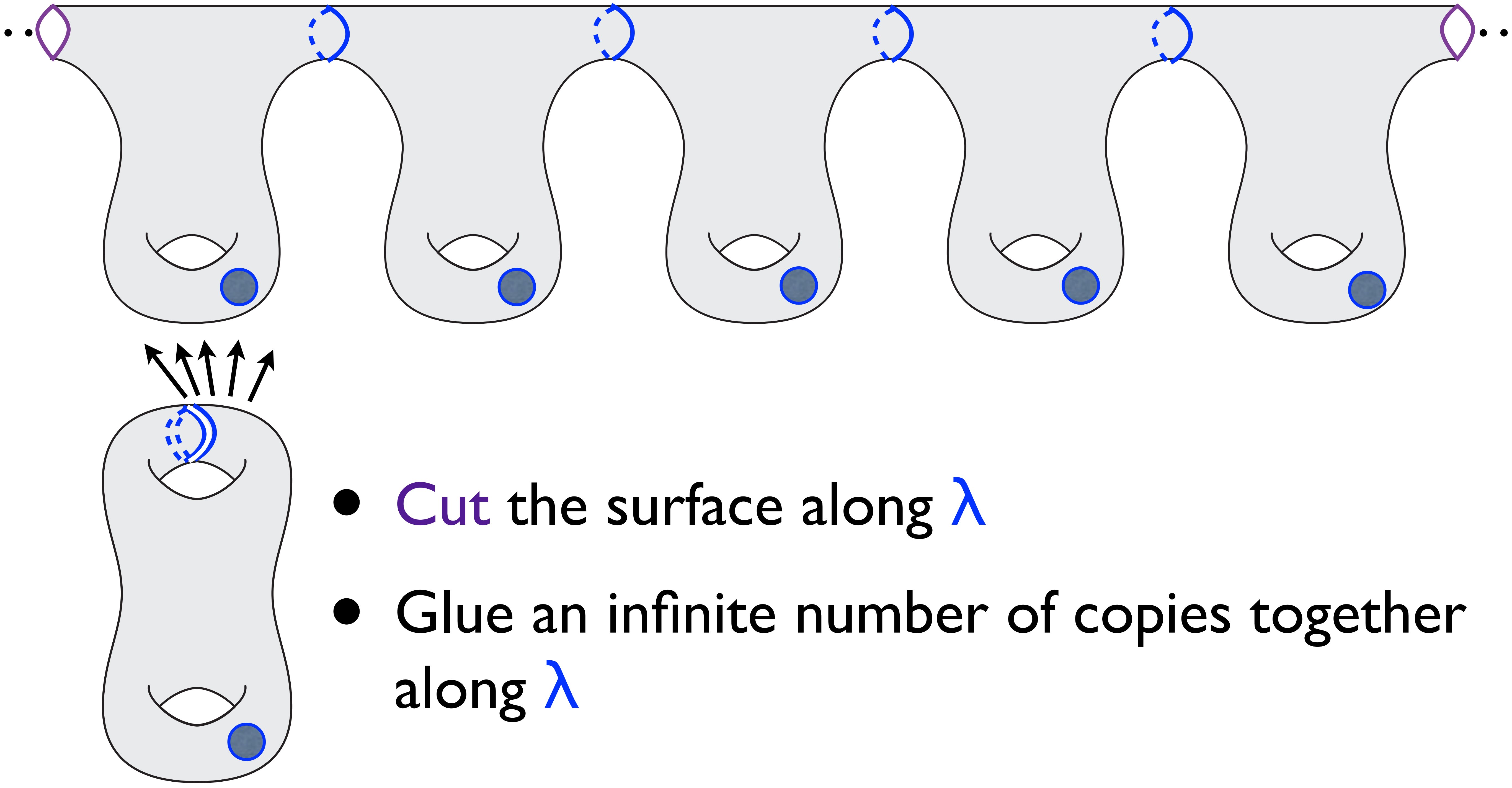
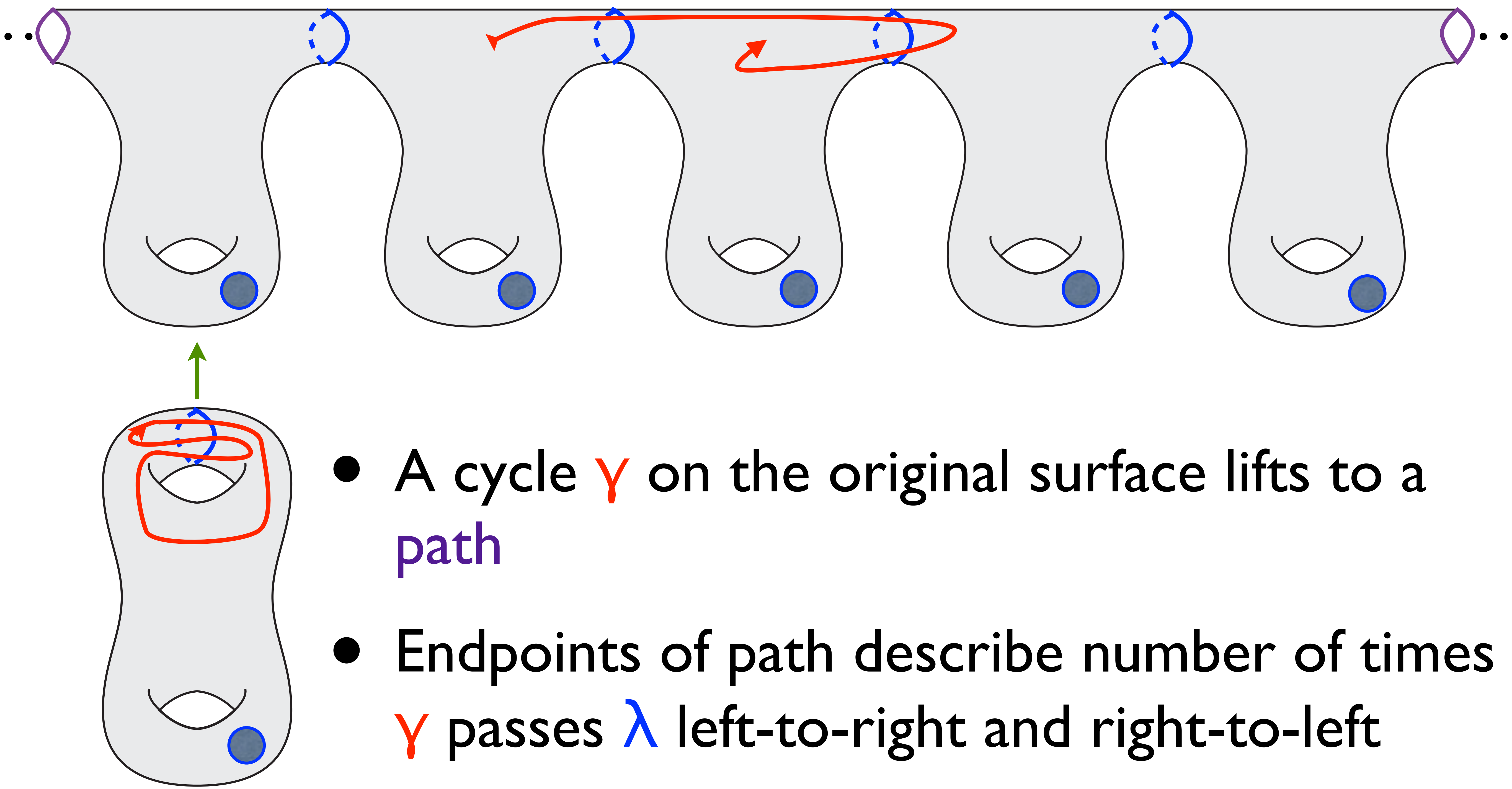


- Cut the surface along λ
- Glue an infinite number of copies together along λ

Cycles in the Cover



- A cycle γ on the original surface lifts to a path
- Endpoints of path describe number of times γ passes λ left-to-right and right-to-left

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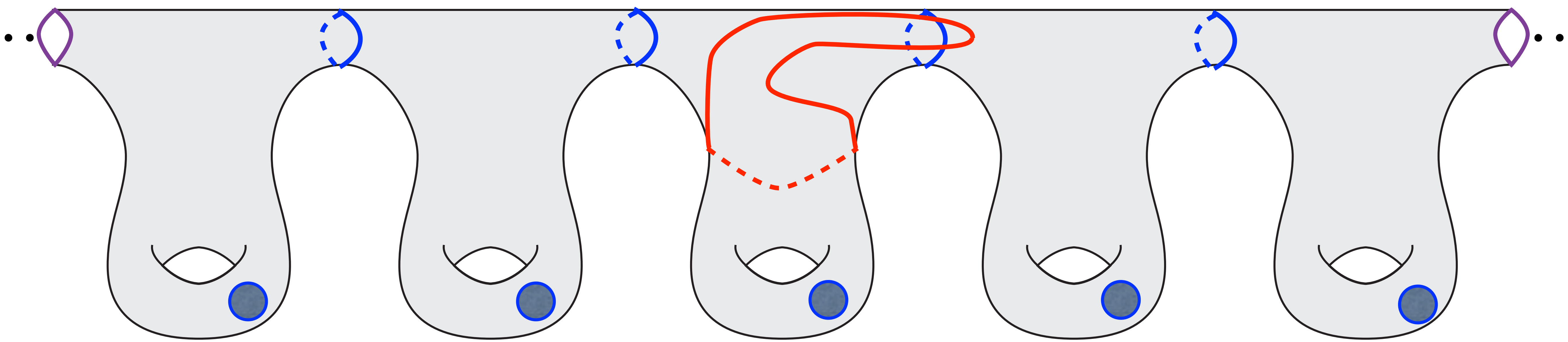

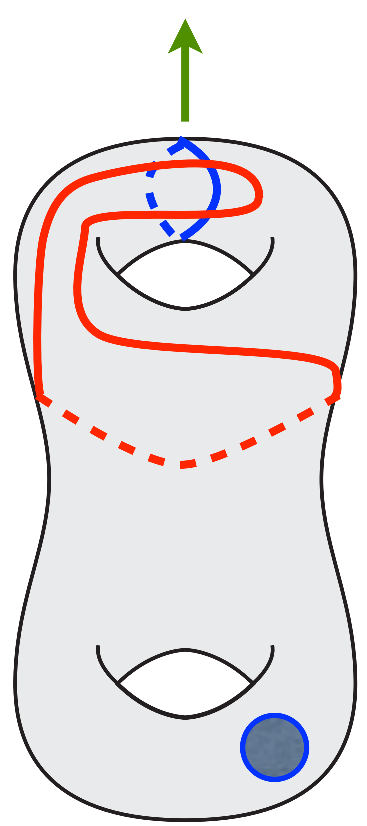

- Cycle γ on the original surface lifts to a cycle if and only if it crosses λ left-to-right the same number of times as it crosses right-to-left
- Any separating cycle lifts to a cycle

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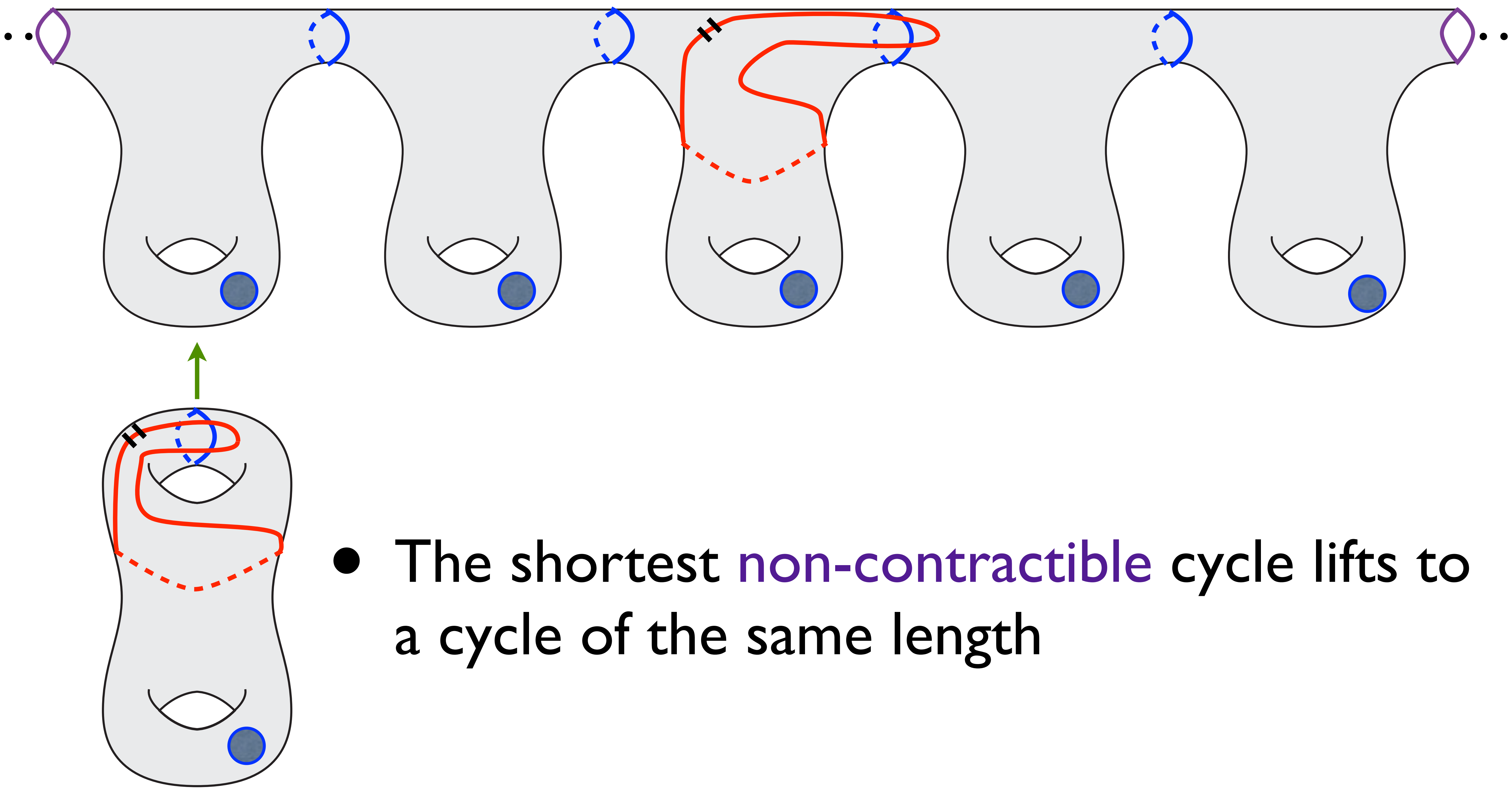
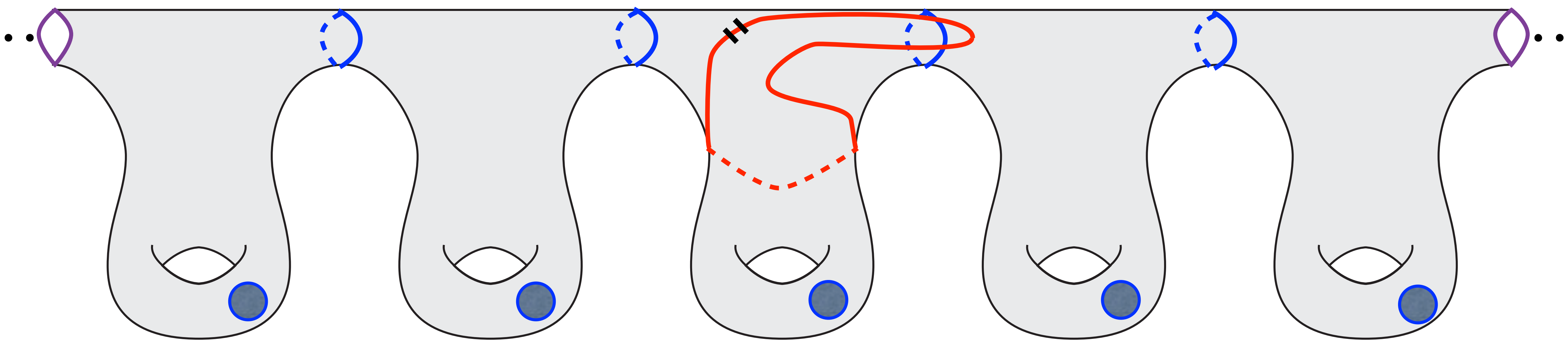
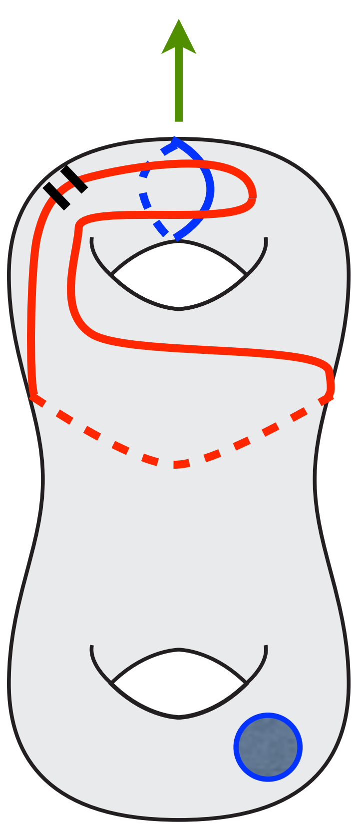


- The shortest non-contractible cycle lifts to a cycle of the same length

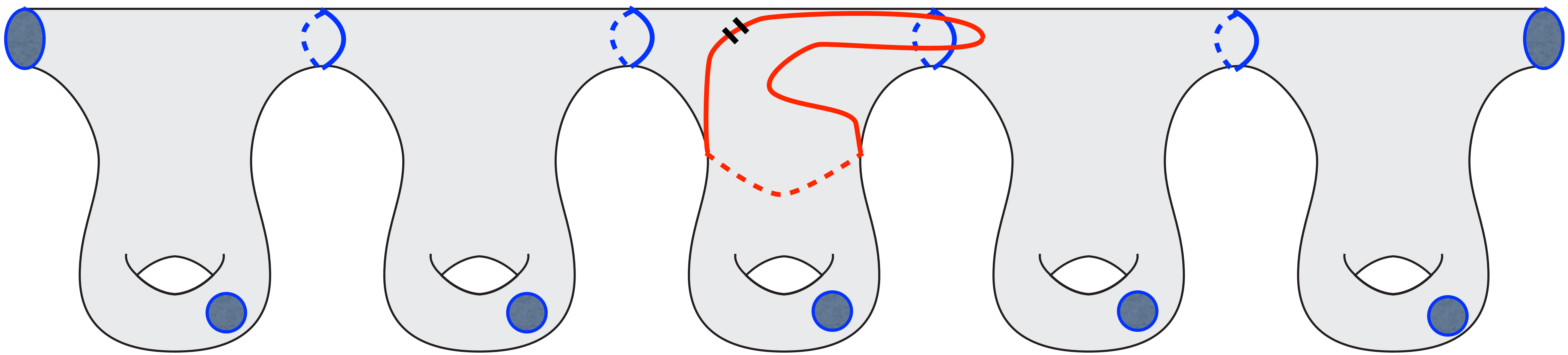
Path Intersections

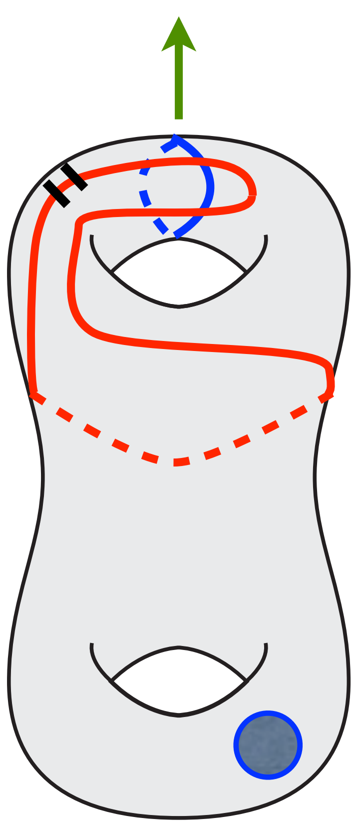




- The shortest non-contractible cycle intersects at most 2 lifts of any shortest path [Erickson '11]

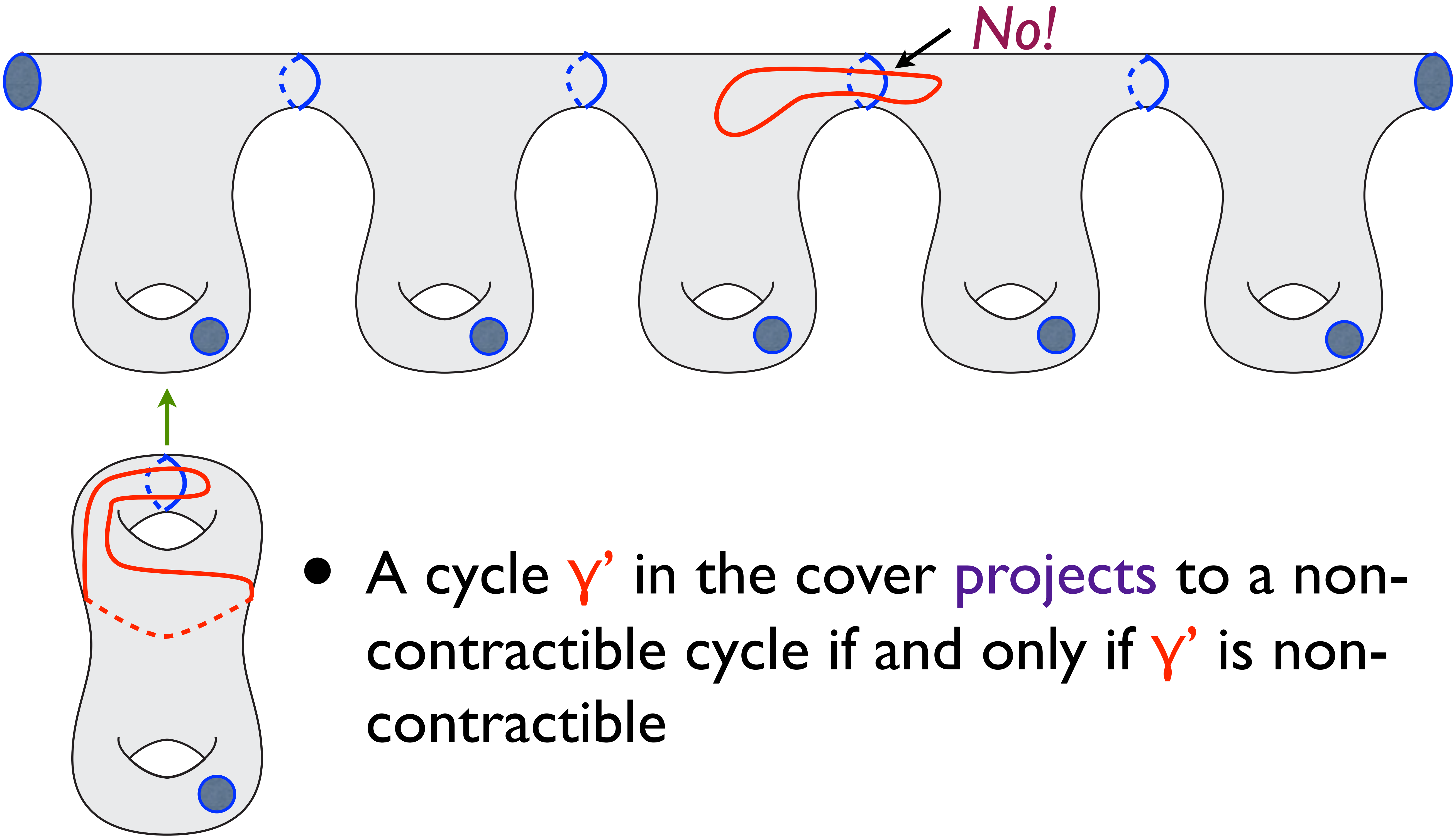
Restricted Infinite Cyclic Cover





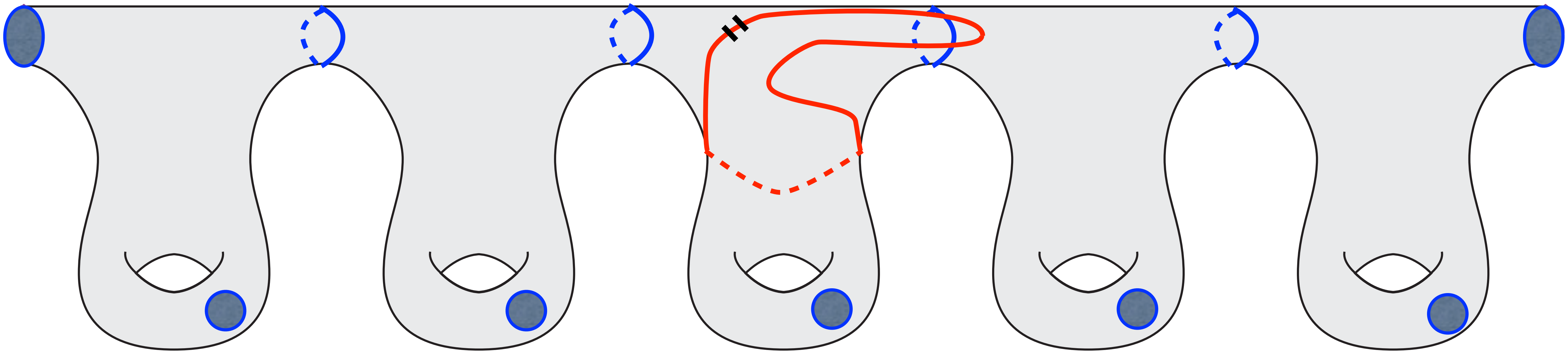
- We only need 5 copies of the original surface in the cover if we cut along a cycle made from two shortest paths
- Leave boundaries at the ends of the left and rightmost copies

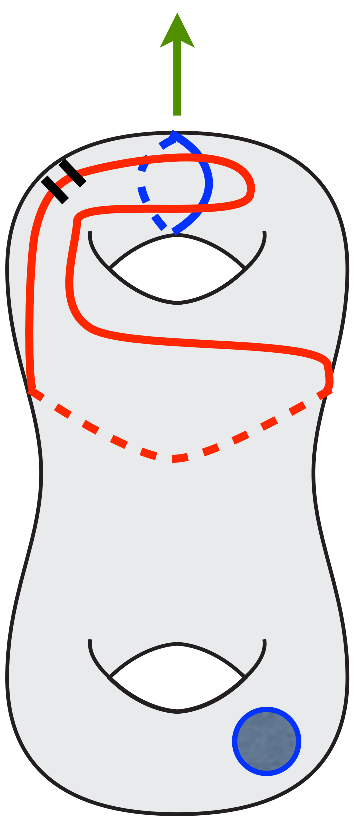
Non-contractible Lift



- A cycle γ' in the cover projects to a non-contractible cycle if and only if γ' is non-contractible

Non-contractible Lift





- The shortest non-contractible cycle in the original surface is the shortest non-contractible cycle in the cover

Recap

- Many non-separating cycles can be used to create the restricted infinite cyclic cover

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- Suffices to find the shortest non-contractible cycle in any subset of the cover

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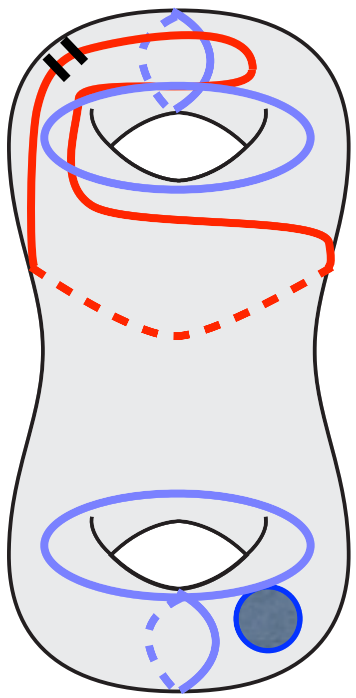
- Suffices to find the shortest non-contractible cycle in any subset of the cover
- But the genus increased!

Separating Boundary



- Compute a system of 2g non-separating cycles from shortest paths in $O(n \log n + g\, n)$ time (a homology basis)

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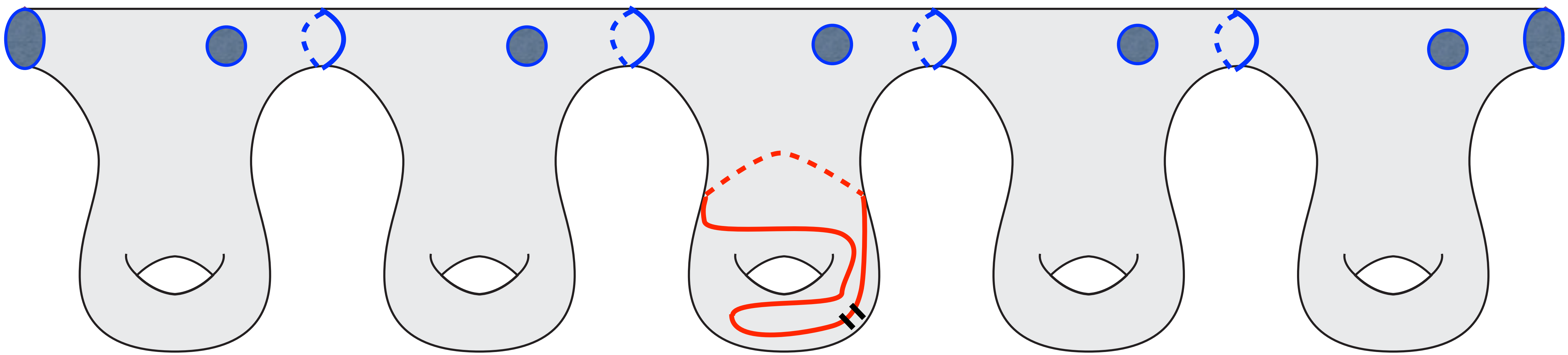

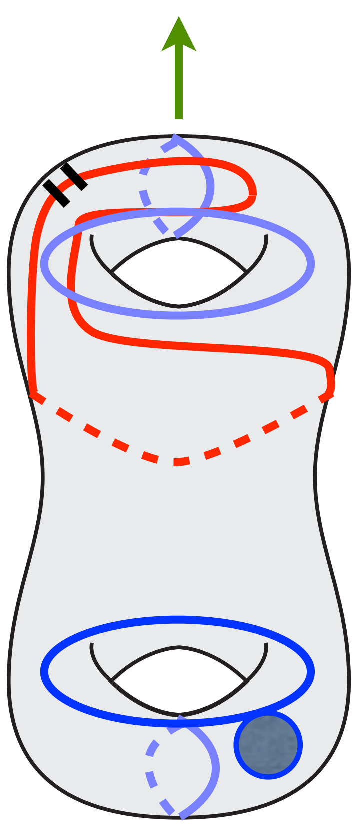

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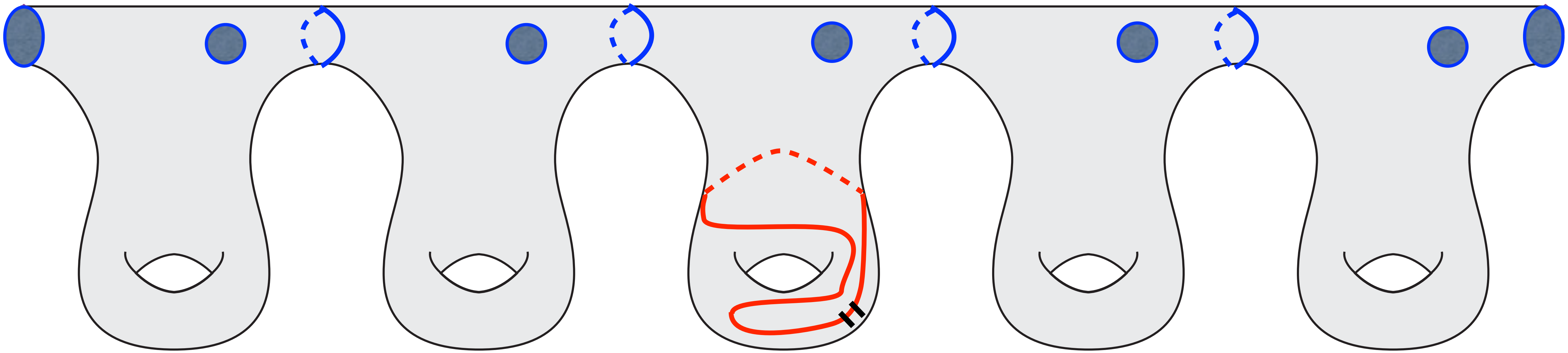

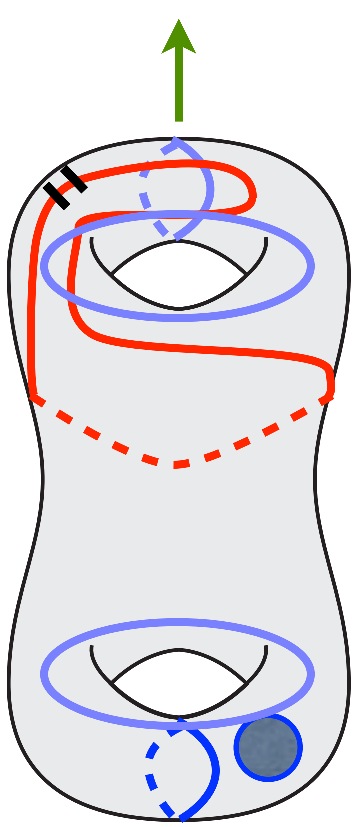

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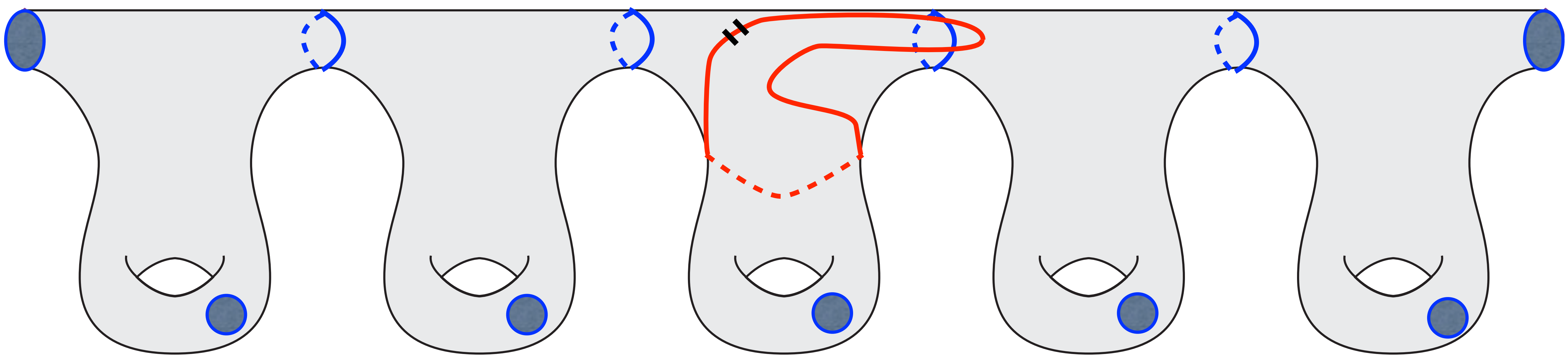

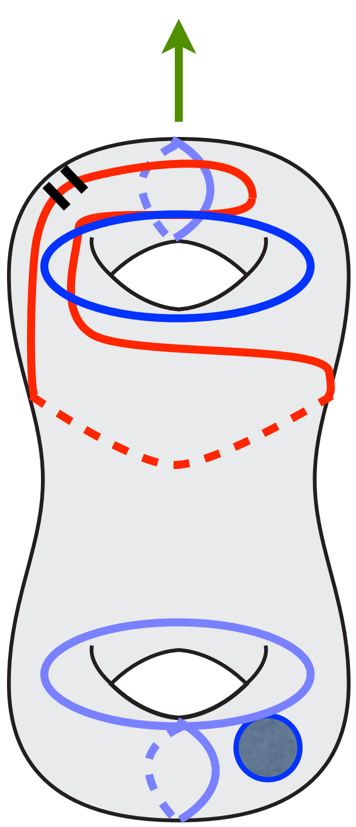

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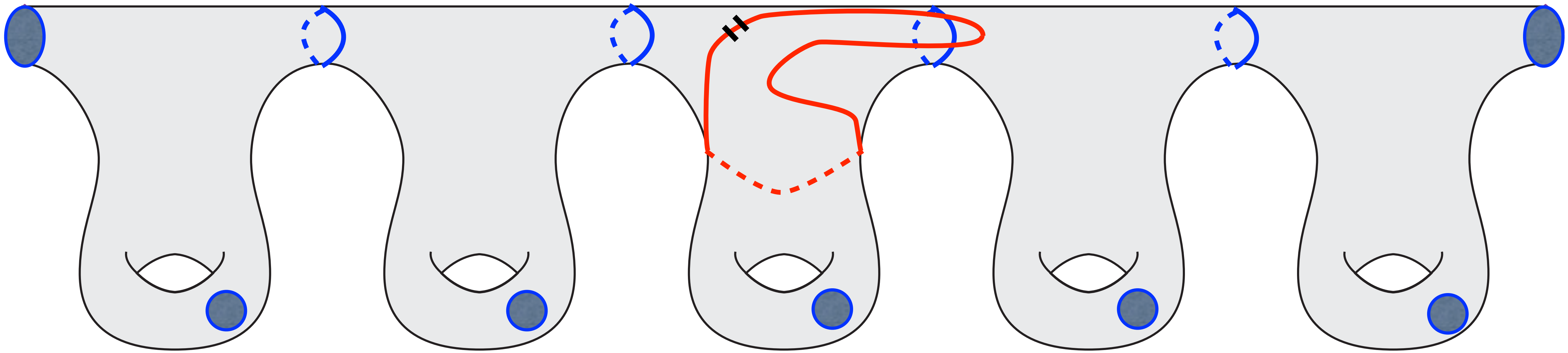
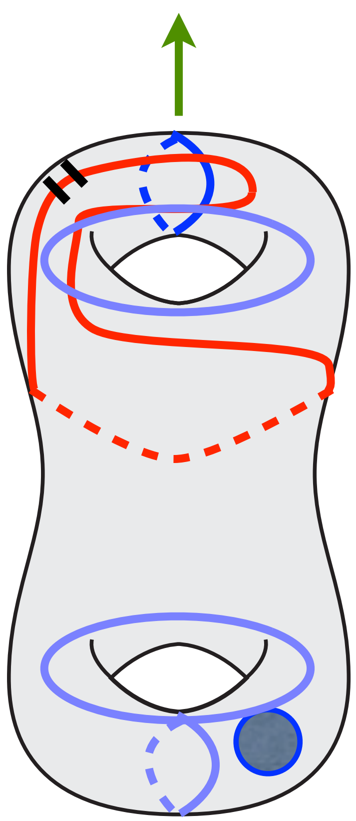
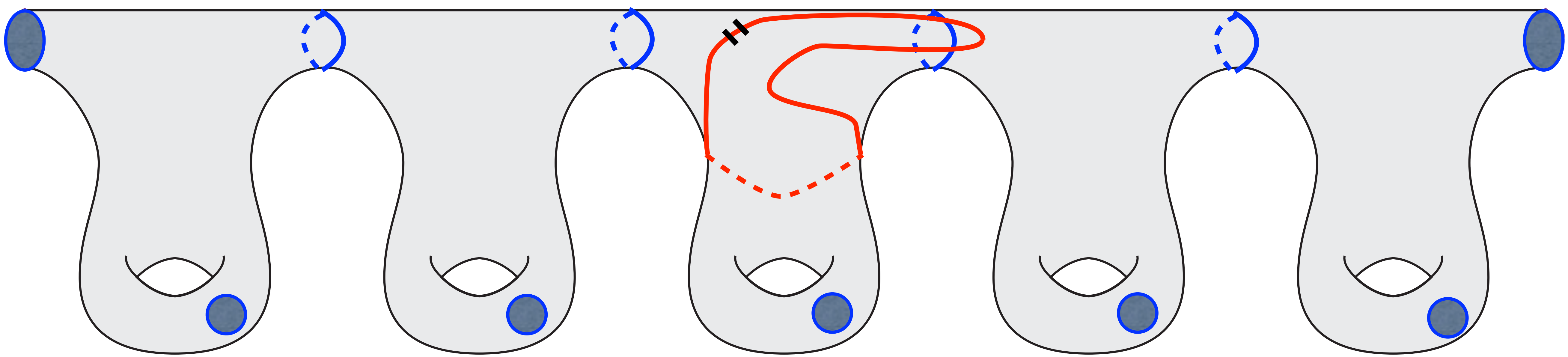
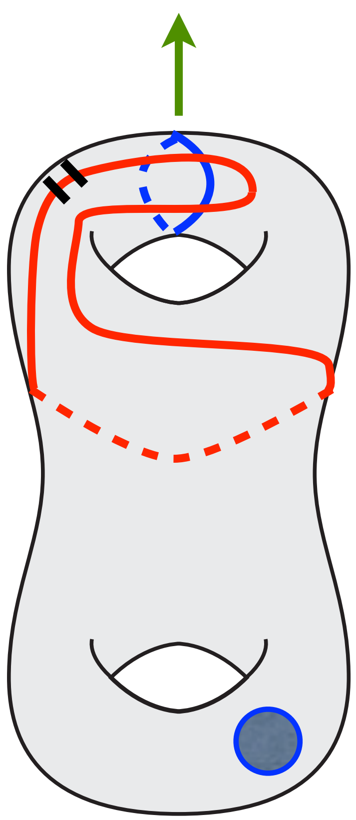




- At least one of the non-separating cycles yields a useful copy of the restricted infinite cyclic cover

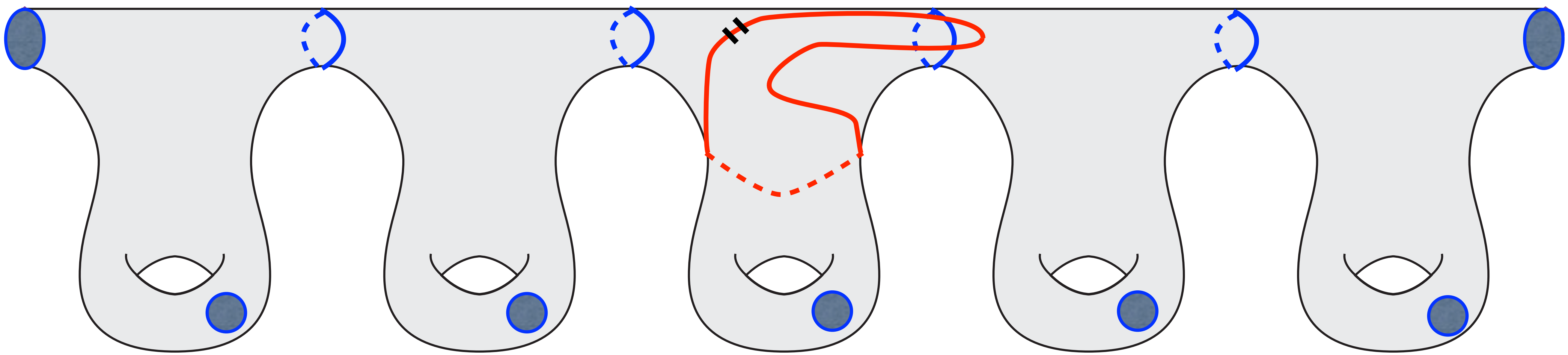
Separating Boundary

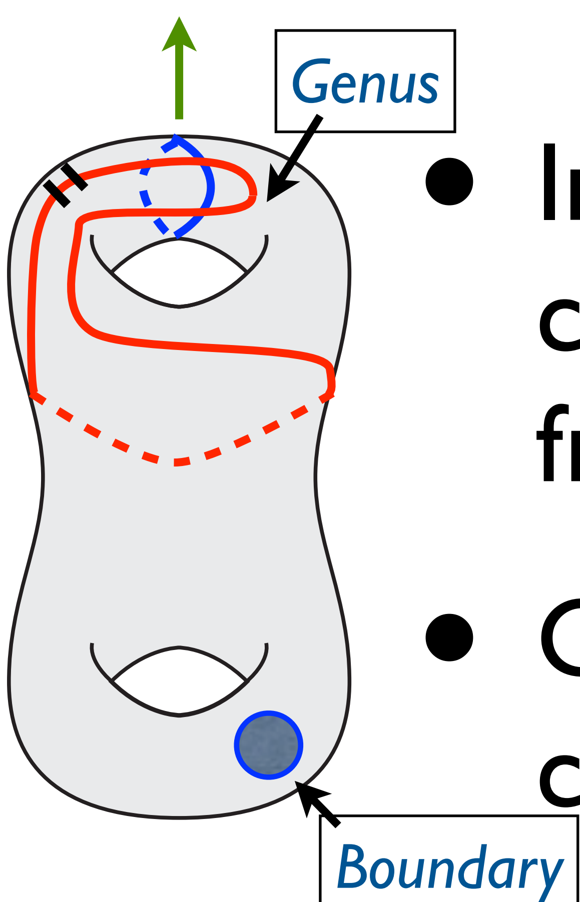




- Main Lemma: At least one lift of the shortest non-contractible cycle is a shortest non-null-homologous cycle
- Lift is non-separating or separates a pair of boundary

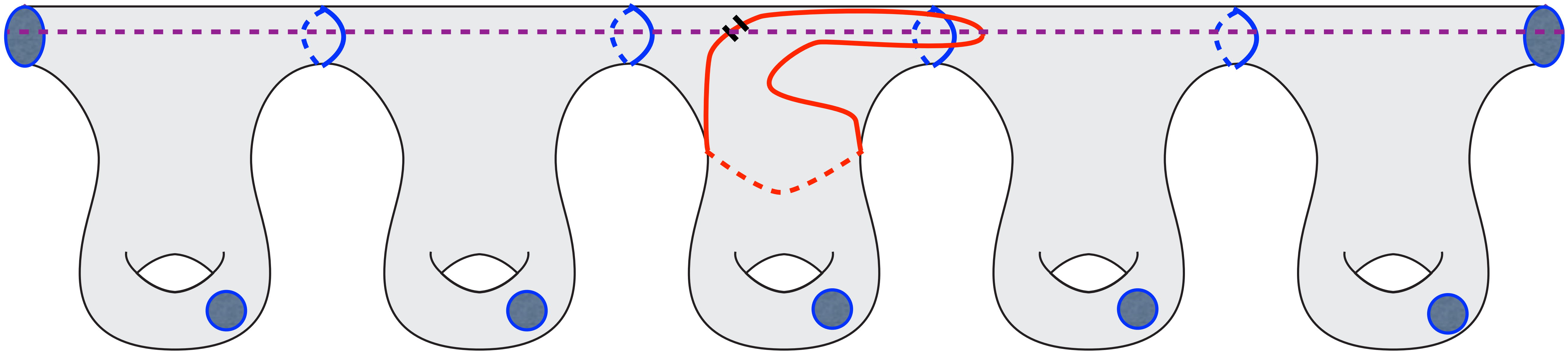
Separating Boundary

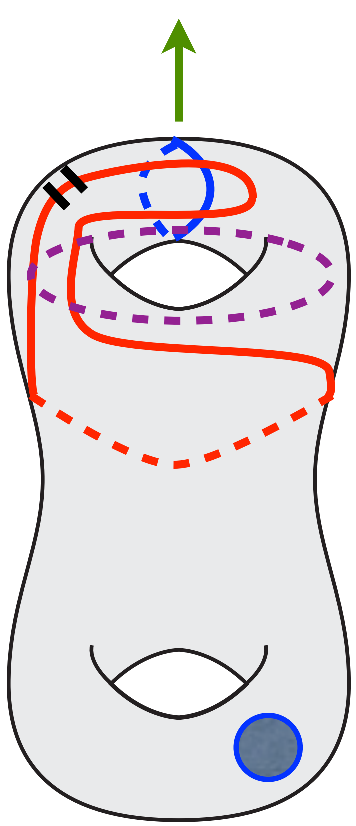




- Intuition: the shortest non-contractible cycle separates the boundary component from a surface subset containing genus
- Genus becomes boundary in the restricted cover

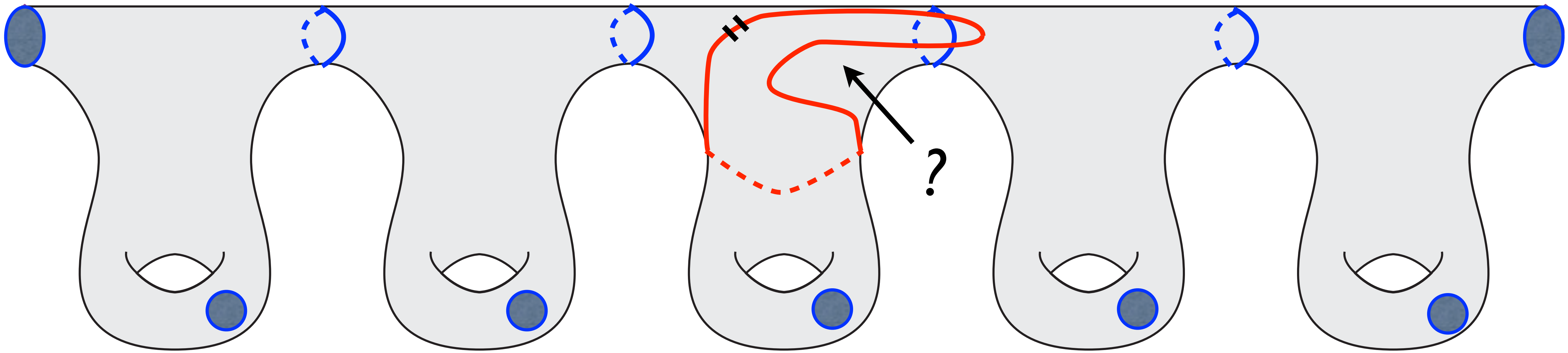
Separating Boundary

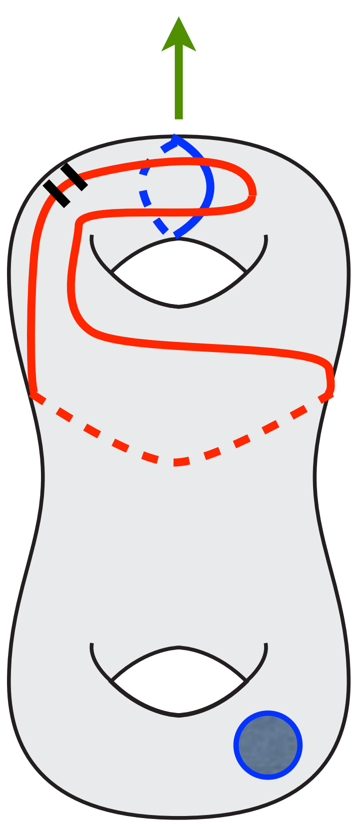




- Theory: A non-separating cycle ω is separated from the boundary
- ω lifts to an arc separated from lift of original boundary

Algorithm: Search the Covers





- In each copy of the restricted infinite cyclic cover, find the shortest cycle that is non-null-homologous

Running Time

- Can search for short cycles in $O(g^2\, n \log n)$ time per covering space
- $O(g^3\, n \log n)$ time spent searching $2g$ covers

Summary of Results

- New algorithms for computing shortest non-trivial cycles in undirected and directed surface graphs
- Included $O(g^3 n \log n)$ time algorithm for shortest non-contractible cycles in directed graphs – first with near-linear dependency on n and sub-exponential dependency on g

Open Problems

- Can we do $O(g^2 \, n \log n)$?
  - Would be easy if there always existed some lift separating boundary
- Other problems in directed graphs
  - Minimum *s*,*t*-cut
  - Minimum quotient cut

Thank you